A one-liner to the critical behavior of the BCS order parameter

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Textbooks on Solid State Physics, such as[2, 3, 4], include a mandatory chapter on Superconductivity. Usually the basic item to start with is the famous Bardeen-Cooper-Schrieffer model(BCS)[1]. One of the main results concerns the way in which the superconducting order parameter $\mathbf{\Delta}(T)$ vanishes at the critical temperature T_c, namely as

$$\mathbf{\Delta}(T) \sim \tilde{a}(T_c - T)^\alpha \tag{1}$$

with the classical exponent $\alpha = 1/2$ and the prefactor $\tilde{a}$.

Then one may read that this is a standard result for any mean-field theory, although the student may wonder, *why it does not follow straightforwardly from the model?*

Yet in the literature this outstandingly simple statement is obtained in a rather roundabout manner. Furthermore α and $\tilde{a}$ are computed only in the weak-coupling limit $\hbar\omega_D \gg k_B T_c$, where ω_D is the Debye frequency. This is certainly an aesthetically not very pleasing situation and I doubt the student really wants to grind through the approximations just to get this simple result.

The following lines show a little trick straightening out this situation. It will hopefully find its way to the textbooks.

In the BCS theory the order-parameter $\mathbf{\Delta}(T)$ satisfies the non-linear integral equation[1]

$$1 = g \int_0^{\hbar\omega_D} d\epsilon \frac{\tanh\left(\frac{\beta E}{2}\right)}{2E}, \tag{2}$$

with $E = \sqrt{\epsilon^2 + \mathbf{\Delta}^2}, \beta = 1/k_B T$ and g is some coupling constant.

We extract the critical behavior of the order parameter straightforwardly and without approximations. For this purpose we choose $\mathbf{\Delta}$ to be real and parametrize it as

$$\mathbf{\Delta}(\beta) = a\left(\frac{\beta - \beta_c}{\beta_c}\right)^\alpha; \ \beta \sim \beta_c. \tag{3}$$

This yields for the derivative $\partial_\beta \mathbf{\Delta}^2 \equiv \frac{\partial \mathbf{\Delta}^2}{\partial \beta}$:

$$\lim_{T \to T_c} \partial_\beta \mathbf{\Delta}^2 = \begin{cases} 0 & \alpha > 1/2 \\ a^2/\beta_c & \alpha = 1/2 \\ \infty & \alpha < 1/2 \end{cases} \tag{4}$$

[1] See e.g [4] equation(23.20) or [3] equation (6.28).

The non-linear integral equation (2) for the order parameter has the solution $\mathbf{\Delta}(\beta, \omega_D, g)$, depending on three parameters. Substituting this solution into equation (2) yields an identity. Differentiating this identity with respect to β easily yields the following relation

$$\partial_\beta \mathbf{\Delta}^2(\beta, \omega_D, g) = \frac{\int_0^{\hbar\omega_D} \frac{d\epsilon}{\cosh^2 \frac{\beta E}{2}}}{\int_0^{\hbar\omega_D} \frac{d\epsilon}{E^3}\Big(\tanh\frac{\beta E}{2} - \frac{\beta E}{2\cosh^2\frac{\beta E}{2}}\Big)}. \tag{5}$$

Taking the limit $T \to T_c, \mathbf{\Delta} \to 0$, we obtain

$$0 < a^2 = \frac{2(k_B T_c)^2 \tanh\frac{\hbar\omega_D\beta_c}{2}}{\int_0^{\hbar\omega_D\beta_c} \frac{dx}{x^3}\Big(\tanh\frac{x}{2} - \frac{x}{2\cosh^2\frac{x}{2}}\Big)} < \infty \tag{6}$$

implying $\boldsymbol{\alpha} = \mathbf{1/2}$. Notice that the above integrand is finite at $x = 0$.

As illustration we evaluate the integral for $\hbar\omega_D\beta_c = 10$ to get

$$\mathbf{\Delta}(T) = 3.10 \cdot k_B T_c \Big(1 - \frac{T}{T_c}\Big)^{\frac{1}{2}},\ T \lesssim T_c. \tag{7}$$

References

[1] J. Bardeen, L. N. Cooper and J. R. Schrieffer, "Microscopic Theory of Superconductivity", Phys. Rev. **106**, 162 (1957).

[2] A.L. Fetter and J.D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill Company, New York, 1971).

[3] A. Atland and B. Simon, *Condensed Matter Field Theory* (Cambridge University Press, Cambridge 2010).

[4] E.C. Marino, *Quantum Field Theory Approach to Condensed Matter Physics* (Cambridge University Press, Cambridge, 2017).