When first studying mechanics one has the impression that everything in this branch of science is simple, fundamental and settled for all time. One would hardly suspect the existence of an important clue which no one noticed for three hundred years. The neglected clue is connected with one of the fundamental concepts of mechanics—that of mass.

A. EINSTEIN AND L. INFELD,
The Evolution of Physics (1938)

12
Inertial forces and noninertial frames

IMAGINE THAT YOU are sitting in a car on a very smooth road. You are holding a heavy package. The car is moving, but you cannot see the speedometer from where you sit. All at once you get the feeling that the package, instead of being just a dead weight on your knees, has begun to push backward horizontally on you as well. Even though the package is not in contact with anything except yourself, the effect is as if a force were being applied to it and transmitted to you as you hold it still with respect to yourself and the car. If you did not restrain the package in this way, it would in fact be pushed backward. You notice that this is what happens to a mascot that has been hanging at the end of a previously vertical string attached to the roof of the car.

How do you interpret these observations? If you have any previous experience of such phenomena, you will have no hesitation in saying that they are associated with an increase of velocity of the car—i.e., with a positive acceleration. Even if this were your first experience of this type, but if you had a well-developed acquaintance with Newton's laws, you could reach the same conclusion. An acceleration of the car calls for an acceleration of everything connected with it; the acceleration of the package requires, through $F = ma$, a force of the appropriate size supplied by your hands. Nonetheless, it does feel just as if the package itself is somehow subjected to an extra force—a "force of inertia"—that comes into play whenever the effort is made to change the state of motion of an object.

These extra forces form an important class. They can be held responsible for such phenomena as the motion of a Foucault pendulum, the effects in a high-speed centrifuge, the so-called g forces on an astronaut during launching, and the preferred direction of rotation of cyclones in the northern and southern hemispheres. These forces are unique, however, in the sense that one cannot trace their origins to some other physical system, as was possible for all the forces previously considered. Gravitational, electromagnetic, and contact forces, for example, have their origins in other masses, other charges, or the "contact" of another object. But the additional forces that make their appearance when an object is being accelerated have no such physical objects as sources. Are these inertial forces real or not? That question, and the answer to it, is bound up with the choice of reference frame with respect to which we are analyzing the motion. Let us, therefore, begin this analysis with a reminder of dynamics from the standpoint of an unaccelerated frame.

MOTION OBSERVED FROM UNACCELERATED FRAMES

An unaccelerated reference frame belongs to the class of reference frames that we have called *inertial*. We saw, in developing the basic ideas of dynamics in Chapter 6, that a unique importance and interest attaches to these frames, in which Galileo's law of inertia holds. We have seen how, if one such frame has been identified, any other frame having an arbitrary constant velocity relative to the first is also inertial, and our inferences about the forces acting on an object are the same in both.

To a good first approximation, as we know, the surface of the earth defines an inertial frame. So also, therefore, does any system moving at constant speed over the earth. Galileo himself was the first person to present a clear recognition of this fact, and one aspect of it that he discussed is useful as a starting point for us now. In his *Dialogue Concerning the Two World Systems*, in which he advocated the Copernican view of the solar system in preference to the Ptolemaic, Galileo pointed out that a rock, dropped from the top of the mast of a ship, always lands just at the foot of the mast, whether or not the ship is moving. Galileo argued from this that the vertical path of a falling object does not compel one to the conclusion that the earth is stationary.

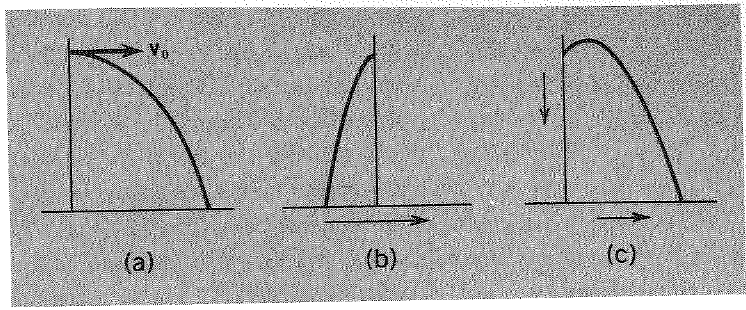


Fig. 12–1 (a) Parabolic trajectory under gravity, as observed in the earth's reference frame. The initial velocity v_0 is horizontal. (b) Same motion observed from a frame with a horizontal velocity greater than v_0. (c) Same motion observed from a frame having both horizontal and vertical velocity components.

The comparison here is between an object falling from rest relative to the earth and another object falling from rest relative to the ship. If we considered only an object that starts from rest relative to a moving ship, its path would be vertical in the ship's frame and parabolic in the earth's frame. More generally, if we considered an object projected with some arbitrary velocity relative to the earth, its subsequent path would have diverse shapes as viewed from different inertial frames (see Fig. 12–1) but all of them would be parabolic, and all of them, when analyzed, would show that the falling object had the vertical acceleration, **g**, resulting from the one force $\mathbf{F}_g$ ($= m\mathbf{g}$) due to gravity. Let us now contrast this with what one finds if the reference frame itself has an acceleration.

MOTION OBSERVED FROM AN ACCELERATED FRAME

Suppose that an object is released from rest in a reference frame that has a constant horizontal acceleration with respect to the earth's surface. Let us consider the subsequent motion as it appears with respect to the earth and with respect to the accelerating frame. We shall take the direction of the positive x axis in the direction of the acceleration and will set up two rectangular coordinate systems: system S, at rest relative to the earth, and S', fixed in the accelerating frame (Fig. 12–2). Take

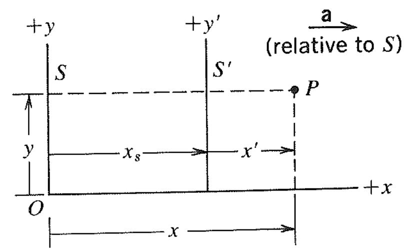


Fig. 12–2 Relationship of coordinates of a particle in two frames that are in accelerated relative motion.

the origins of the frames to coincide at $t = 0$, and suppose that the velocity of S' with respect to S at this instant is equal to v_0. The vertical axes of the two systems are taken as positive upward, and the object is released at $t = 0$ from a point for which $x = x' = 0$, $y = y' = h$.

What will the trajectories in S and S' look like? For an observer in S, we already know the answer. To him, the object is undergoing free fall with initial horizontal velocity v_0 [Fig. 12–3(a)]. Thus we have

$$\text{(As observed in } S) \begin{cases} x = v_0 t \\ y = h - \tfrac{1}{2}gt^2 \end{cases}$$

These two equations uniquely define the position of the object at time t, but to describe the motion as observed in S' we must express the results in terms of the coordinates x' and y' as measured in S'. To transform to the S' frame, we substitute

$$x' = x - x_s$$
$$y' = y$$

where x_s is the separation along the x axis of the origins of S and S' (see Fig. 12–2). We know that

$$x_s = v_0 t + \tfrac{1}{2}at^2$$

Substituting these values we find

$$\text{(As observed in } S') \begin{cases} x' = v_0 t - (v_0 t + \tfrac{1}{2}at^2) = -\tfrac{1}{2}at^2 \\ y' = h - \tfrac{1}{2}gt^2 \end{cases}$$

Thus the path of the particle as observed in S' is a *straight line* given by the equation

$$x' = -\frac{a}{g}(h - y')$$

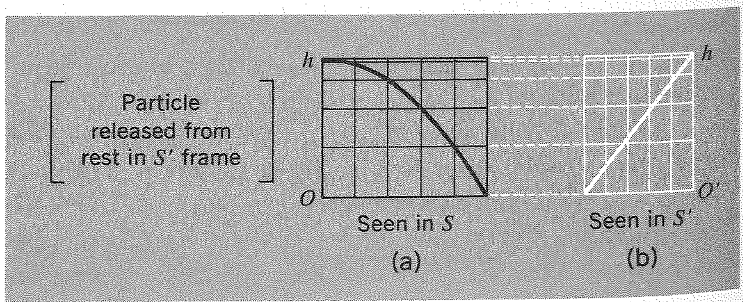


Fig. 12–3 (a) Parabolic trajectory of a particle under gravity, as observed in the earth's reference frame S. (b) Same motion observed in a frame S′ that has a constant horizontal acceleration.

This is shown in Fig. 12–3(b). In the accelerated frame, the object appears to have not only a constant downward component of acceleration due to gravity, but also a constant horizontal component of acceleration in the $-x$ direction which causes the particle to follow a nonvertical straight-line path. [A similar simple example is the monkey-shooting drama described in Chapter 3 (p. 104). The outcome becomes almost self-evident if we choose to describe the events in the rest frame of the falling monkey. In this frame the bullet just follows a straight-line path directly toward the monkey, while the ground accelerates upward at 9.8 m/sec^2.]

There is no mystery about the unfamiliar motion represented by Fig. 12–3(b). It is a direct kinematic consequence of describing the normal free-fall motion from a frame that is itself accelerated. We could perfectly well use this path, described by measurements made entirely within S', to discover the acceleration of this frame, provided that the direction of the true vertical were already known. However, a greater interest attaches to learning about the acceleration through dynamic methods. That is the concern of the next section.

ACCELERATED FRAMES AND INERTIAL FORCES

From what has been said, it is clear that inertial frames have a very special status. All inertial frames are *equivalent* in the sense that it is impossible by means of dynamical experiments to discover their motions in any absolute sense—only their relative motions are significant. Out of this dynamical equivalence comes what is called the Newtonian principle of relativity:

> **There is no dynamical observation that leads us to prefer one inertial frame to another. Hence, no dynamical experiment will tell us whether we have a constant velocity through space.**

As we have just seen, however, a relative acceleration between two frames *is* dynamically detectable. As observed in accelerating frames, objects have unexpected accelerations. It follows at once, since Newton's law establishes a link between force and acceleration, that we have a quantitative basis for calculating the magnitude of the inertial force associated with a measured acceleration. Conversely, and more importantly, we have a dynamical basis for inferring the magnitude of an acceleration from the inertial force associated with it. This is the underlying principle of all the instruments known as accelerometers. They function because of the inertial property of some physical mass.

To make the analysis explicit, consider the motion of a particle P with respect to two reference frames like those considered in the last section and shown in Fig. 12–2: an inertial frame S and an accelerated frame S'. We then have, once again,

$$x = x' + x_s$$
$$y = y'$$

The velocity components of P as measured in the two frames are thus given by

$$u_x = u'_x + v_s$$
$$u_y = u'_y$$

where $v_s = dx_s/dt$ at any particular instant. If S' has a constant acceleration a, we can put $v_s = v_0 + at$, but the condition of constant acceleration is not at all necessary to our analysis.

Taking the time derivatives of the instantaneous velocity components, we then get

$$a_x = a'_x + a_s$$
$$a_y = a'_y$$

where a_s is the instantaneous acceleration of the frame S'. Although we have chosen to introduce the calculation in terms of Cartesian components, it is clear that a single vector statement relates the acceleration $\mathbf{a}$ of P, as measured in S, to its acceleration $\mathbf{a}'$ as measured in S' together with the acceleration $\mathbf{a}_s$ of S' itself:

$$\mathbf{a} = \mathbf{a}' + \mathbf{a}_s \qquad (12\text{–}1)$$

Multiplying Eq. (12–1) throughout by m, we recognize the left-hand side as giving the real (net) force, $\mathbf{F}$, that is acting on the particle, since this defines the true cause of its acceleration as measured in an inertial frame. That is, in the S frame,

$$\mathbf{F} = m\mathbf{a} \qquad (12\text{–}2)$$

but, using Eq. (12–1), this gives us

$$\mathbf{F} = m\mathbf{a}' + m\mathbf{a}_s \qquad (12\text{–}3)$$

We now come to the crucial question: How do we interpret Eq. (12–3) from the standpoint of observations made within the accelerated frame S' itself?

Newton's viewpoint— that the net force on an object is the cause of accelerated motion ($\mathbf{F}_{\text{net}} = m\mathbf{a}$)—is so deeply ingrained in our thinking that we are strongly motivated to preserve this relationship at all times. When we observe an object accelerating, we interpret this as due to the action of a net force on the object. Can we achieve a mathematical format that has the appearance of $\mathbf{F}_{\text{net}} = m\mathbf{a}$ for the present case of an accelerated frame of reference? Yes. By transferring all terms but $m\mathbf{a}'$ to the left and treating these terms as forces that act on m, and have a resultant $\mathbf{F}'$, which is of the correct magnitude to produce just the observed acceleration $\mathbf{a}'$:

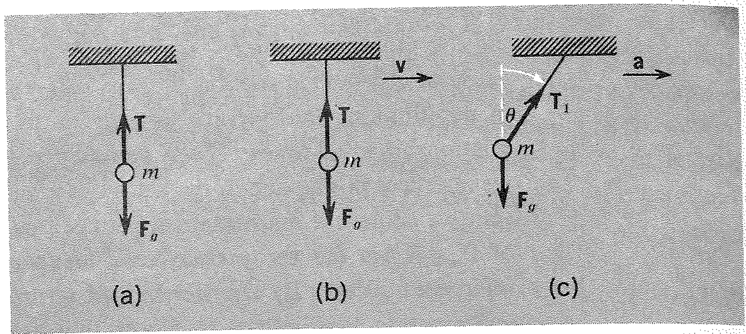
$$\mathbf{F}' = \mathbf{F} - m\mathbf{a}_s = m\mathbf{a}' \qquad (12\text{–}4)$$

The net force in the S' frame is thus made up of two parts: a "real" force, $\mathbf{F}$, with components F_x and F_y, and a "fictitious" force equal to $-m\mathbf{a}_s$, which has its origin in the fact that the frame of reference itself has the acceleration $+\mathbf{a}_s$. An important special case of Eq. (12–4) is that in which the "real" force $\mathbf{F}$ is zero, in which case the particle, as observed in S', moves under the action of the inertial force $-m\mathbf{a}_s$ alone.

The result expressed by Eq. (12–4) is not merely a mathematical trick. From the standpoint of an observer in the accelerating frame, the inertial force is actually present. If one took steps to keep an object "at rest" in S', by tying it down with springs, these springs would be observed to elongate or contract in such a way as to provide a counteracting force to balance the inertial force. To describe such a force as "fictitious" is therefore somewhat misleading. One would like to have some convenient label that distinguishes inertial forces from forces that arise from true physical interactions, and the term "pseudo-force" is often used. Even this, however, does not do justice to such forces as experienced by someone who is actually in the accelerating frame. Probably the original, strictly technical name, "inertial force," which is free of any questionable overtones, remains the best description.

As an illustration of the way in which the same dynamical situation may be described from the different standpoints of an inertial frame, on the one hand, and an accelerated frame, on the other, consider a simple pendulum suspended from the roof of a car. The mass of the bob is m. In applying $\mathbf{F} = m\mathbf{a}$ from the standpoint of a frame of reference S attached to the earth

Fig. 12–4 Forces acting on a suspended mass in (a) a stationary car, (b) a car moving at constant velocity, and (c) a car undergoing a positive acceleration.



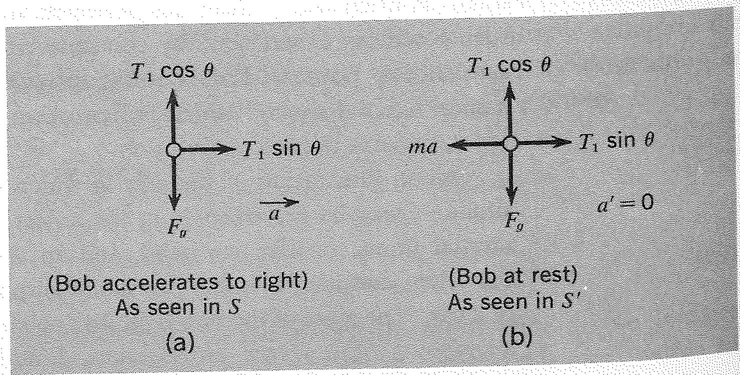
(assumed nonrotating), one can draw isolation diagrams for the possible motions of the car as shown in Fig. 12–4. In each case, there are just two (real) forces acting on the bob: $\mathbf{F}_g$, the force of gravity, and **T**, the tension in the string. Cases (a) and (b) do not involve acceleration and the application of $\mathbf{F} = m\mathbf{a}$ is trivial. In (c), the bob undergoes acceleration toward the right and the string hangs at an angle with some increase in its tension (from T to T_1). The isolation diagram of Fig. 12–5(a) leads us to apply $\mathbf{F} = m\mathbf{a}$ as follows:

Horizontal component: $T_1 \sin\theta = ma$

Vertical component: $T_1 \cos\theta - mg = 0$

In the S' frame, however, because of the acceleration of the frame, there will be an additional force of magnitude ma in the direction opposite to the acceleration of the frame. Figure 12–5(b) shows an isolation diagram for the bob as seen in S'. The bob is in equilibrium. Here, application of $\mathbf{F}' = m\mathbf{a}'$ gives (because $\mathbf{a}' = 0$):

Fig. 12–5 Forces on an object that is at rest relative to an accelerated car (a) as judged in an inertial frame, and (b) as judged in the accelerated frame.



$$T_1 \sin\theta - ma = 0$$
$$T_1 \cos\theta - mg = 0$$

Thus the equilibrium inclination of the pendulum is defined by the condition

$$\tan\theta = \frac{a}{g} \qquad (12\text{–}5)$$

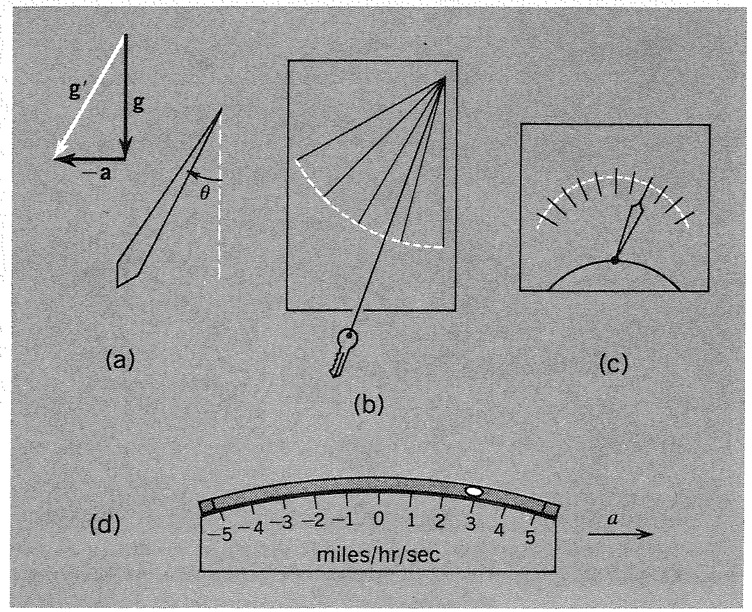
ACCELEROMETERS

The result expressed by Eq. (12–5) provides the theoretical basis for a simple accelerometer. If we have first established the true vertical direction, representing $\theta = 0$, the observation of the angle of inclination of a pendulum at any subsequent time tells us the value of a through the equation

$$a = g \tan\theta$$

For example, if a passenger in an airplane lets his tie, or a key-chain, hang freely from his fingers during the takeoff run, he can make a rough estimate of the acceleration, which is usually almost constant [Fig. 12–6(a)]. If he also records the time from the beginning of the run to the instant of takeoff, he can obtain

Fig. 12–6 (a) Tie hanging in equilibrium within an accelerated vehicle. (b) Quantitative accelerometer based on measuring the equilibrium angle of a simple plumbline. (c) Carpenter's level (in this case a pivoted marker immersed in liquid of greater density) can be used as an accelerometer. (d) A bubble trapped in a curved tube of liquid gives direct readings of acceleration. This form of accelerometer was devised by W. U. Walton (Education Research Center, M.I.T.). Figure 12–7 shows an example of its use.



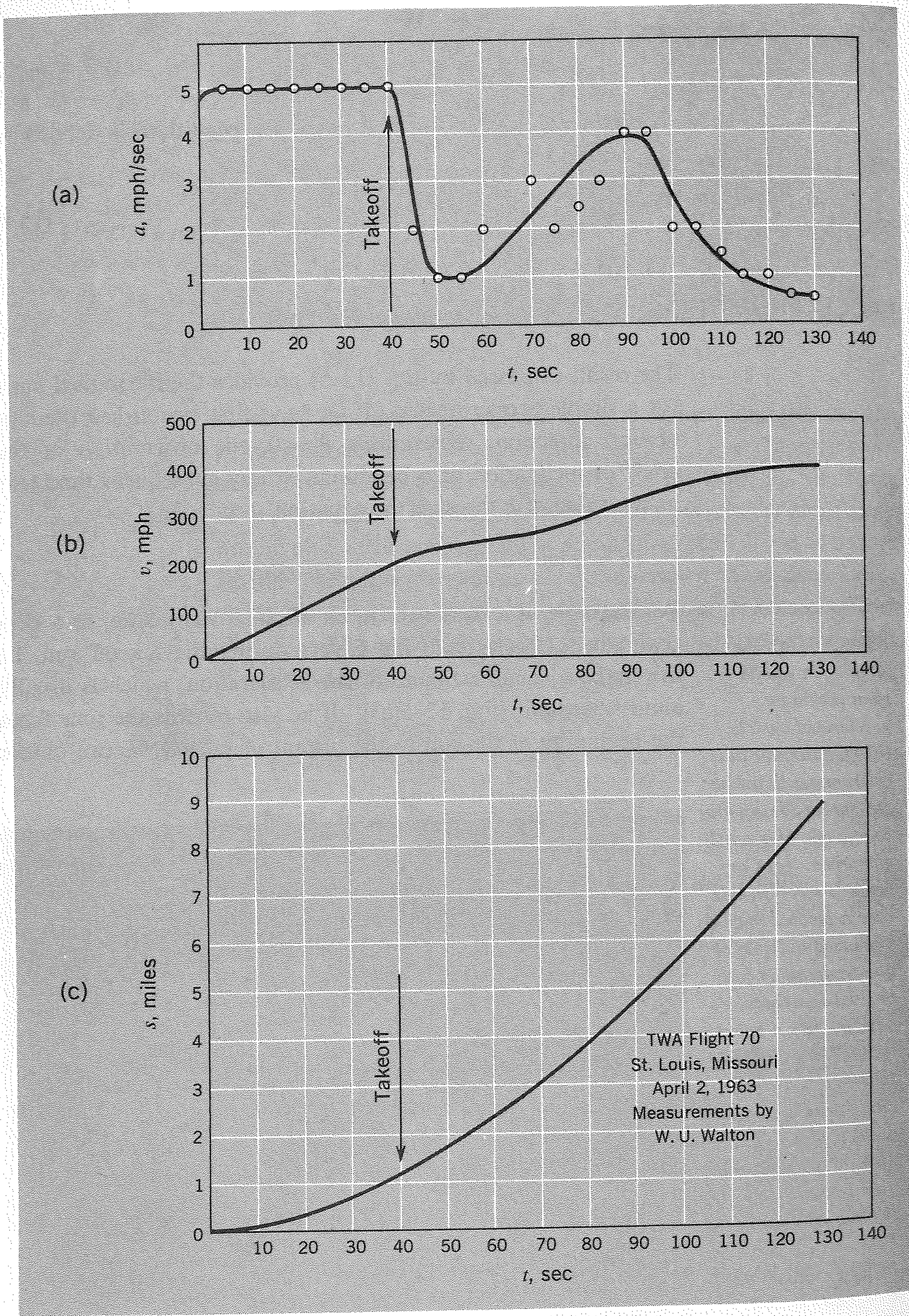


Fig. 12–7 (a) Record obtained with the accelerometer of Fig. 12–6(d) before and after takeoff of a commercial jet aircraft. The accelerometer was held so as to record the horizontal component of acceleration only. Note the sharp

a fairly good estimate of the length of the run and the takeoff speed. If he is more ambitious, he can go armed with a card, as in Fig. 12–6(b), already marked out as a goniometer (= angle measurer) or even directly calibrated in terms of acceleration measured in convenient units (e.g., mph per second).[1] Another simple accelerometer is obtainable readymade in the form of a carpenter's level made of a small pivoted float that is completely immersed in a liquid [Fig. 12–6(c)]. All these devices make use of the fact that the natural direction of a plumbline in an accelerated frame is defined by the combination of the gravitational acceleration vector **g** and the negative of the acceleration **a** of the frame itself.

A quite sensitive accelerometer of this same basic type, with the further advantages of a quick response and a quick attainment of equilibrium (without much overshoot or oscillation) can be made by curving a piece of plastic tubing into a circular arc and filling it with water or acetone until only a small bubble remains [see Fig. 12–6(a)]. Figure 12–7(a) shows the record of acceleration versus time as obtained with such an accelerometer during the takeoff of a jet aircraft. Figures 12–7(b) and (c) show the results of numerically integrating this record so as to obtain the speed and the total distance traveled.

Accelerometers of a vastly more sophisticated kind can be made by using very sensitive strain gauges, with electrical measuring techniques, to record in minute detail the deformations of elastic systems to which a mass is attached. Figure 12–8 shows in schematic form the design of such an instrument. If the object on which the accelerometer is mounted undergoes an acceleration, the inertial force experienced by the pendulum bob begins to deflect it. This, however, unbalances slightly an electrical capacitance bridge in which the pendulum forms part of two of the capacitors, as shown. An error signal is obtained which is used both to provide a measure of the acceleration and to drive a coil that applies a restoring force to the pendulum. Such an accelerometer unit may have a useful range from about 10^{-5} "g" to more than $10\,g$.

[1] A book entitled *Science for the Airplane Passenger* by Elizabeth A. Wood (Ballantine, New York, 1969) has such a goniometer on its back cover and a discussion of its use in the text. The book also describes a host of other ways in which airplane passengers can discover or apply scientific principles during their travels.

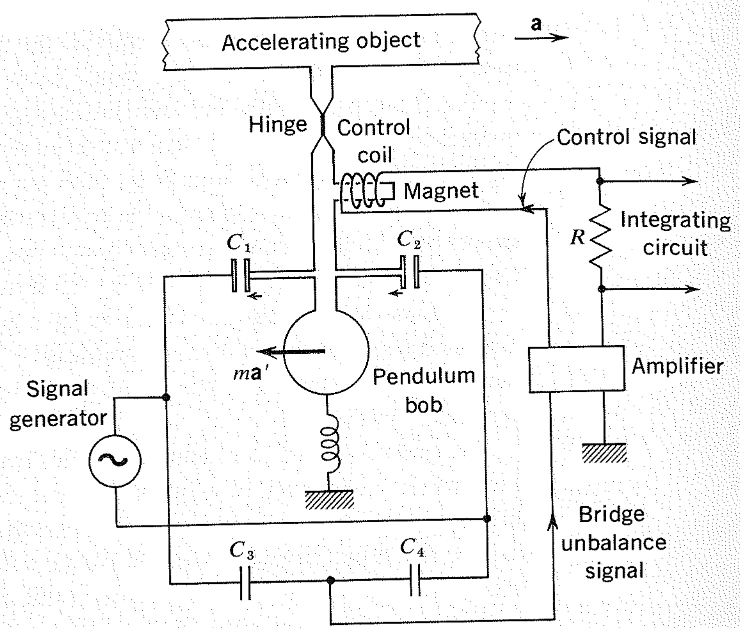


Fig. 12–8 Electromechanical accelerometer system.

ACCELERATING FRAMES AND GRAVITY

In all our discussions of accelerated frames, we have assumed that the observers know "which way is up"—i.e., they know the direction and magnitude of the force of gravity and treat it (as we have done) as a real force, whose source is the gravitating mass of the earth. But suppose our frame of reference to be a completely enclosed room with no access to the external surroundings. What can one then deduce about gravity and inertial forces through dynamical experiments wholly within the room?

We shall suppose once again that there is an observer in a frame, S, attached to the earth. This observer is not isolated; he is able to verify that the downward acceleration of a particle dropped from rest is along a line perpendicular to the earth's surface and hence is directed toward the *center* of the earth.[1] He is able to draw the orthodox conclusion that this acceleration is due to the gravitational attraction from the large mass of the earth. Our second observer is shut up in a room that defines the frame S'. Initially it is known that the floor of his room is horizontal and that its walls are vertical. In subsequent measurements, however, the observer in S' finds that a plumbline hangs at an angle to what he had previously taken to be the vertical, and that objects dropped from rest travel parallel to his plumbline. The observers in S and S' report their findings to one another by radio. The observer in S' then concludes that he has three alternative ways of accounting for the component of force, parallel to the floor, that is now exerted on all particles as observed in *his* frame:

1. In addition to the gravitational force, there is an inertial force in the $-x$ direction due to the acceleration of his frame in the $+x$ direction.
2. His frame is not accelerating, but a large massive object has been set down in the $-x$ direction outside his closed room, thus exerting an additional gravitational force on all masses in his frame.
3. His room has been tilted through an angle θ *and* an extra mass has been placed beneath the room to increase the net gravitational force. (This is close to being just a variant of alternative 2.)

In supposing that all three hypotheses work equally well to explain what happens in S', we must assume that the additional massive object, postulated in alternatives 2 and 3, produces an effectively uniform gravitational field throughout the room.

From dynamical experiments made entirely within the closed room, there is no way to distinguish among these hypotheses. The acceleration of the frame of reference produces effects that are identical to those of gravitational attraction. Inertial and gravitational forces are both proportional to the mass of the object under examination. The procedures for detecting and measuring them are identical. Moreover, they are both describable in terms of the properties of a *field* (an acceleration field) that has a certain strength and direction at any given point. An object placed in this field experiences a certain force without benefit of any contact with its surroundings. Is all this just an interesting parallel, or does it have a deeper significance?

Einstein, after pondering these questions, concluded that there was indeed something fundamental here. In particular, the completely exact proportionality (as far as could be determined) between gravitational force and inertial mass suggested to him that no physical distinction could be drawn, at least within a

[1] We are still ignoring the rotation of the earth, which causes this statement to be not quite correct. A falling object does *not* fall exactly parallel to a plumbline. We shall come back to this when we discuss rotating reference frames.

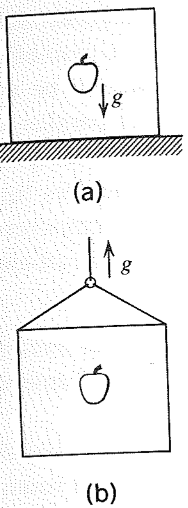
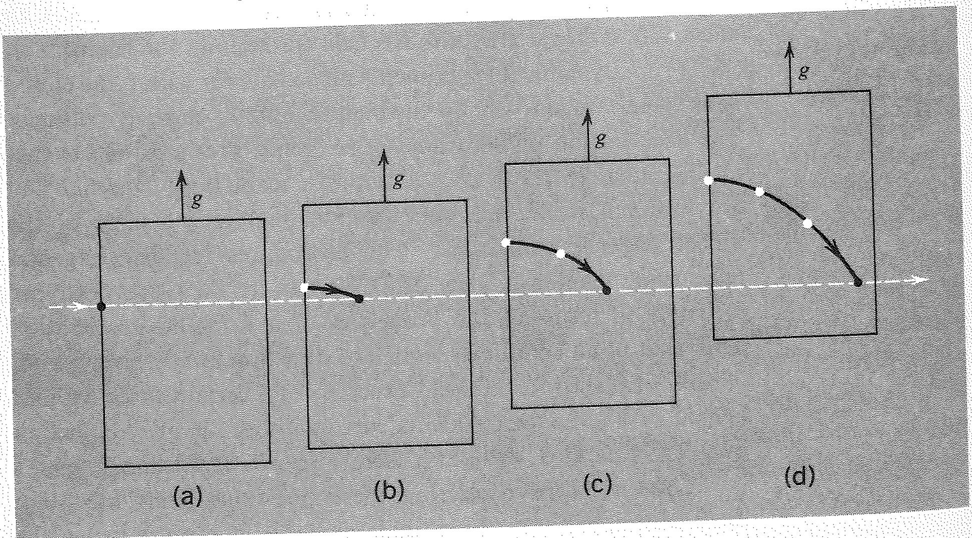


Fig. 12–9 (a) Apple falling inside a box that rests on the earth. (b) Indistinguishable motion when the apple is inside an accelerated box in outer space.

limited region, between a gravitational field and a general acceleration of the reference frame (see Fig. 12–9). He announced this—his famous principle of equivalence—in 1911.[1] The proportionality of gravitational force to inertial mass now becomes an exact necessity, not an empirical and inevitably approximate result. It is also implied that *anything* traversing a gravitational field must follow a curved path, because such a curvature would appear on purely kinematic and geometrical grounds if we replaced the gravitational field by the equivalent acceleration of our own reference frame. In particular, this should happen with rays of light (see Fig. 12–10). With the help of these ideas Einstein proceeded to construct his general theory of relativity, which (as we pointed out in Chapter 8) is primarily a geometrical theory of gravitation.

Fig. 12–10 Successive stages in the path of a horizontally traveling object as observed within an enclosure accelerating vertically upward. This illustrates the equivalence of gravity and a general acceleration of the reference frame.



[1]A. Einstein, *Ann. Phys.* (4) **35**, 898 (1911), reprinted in translated form in *The Principle of Relativity* (W. Perrett and G. B. Jeffery, translators), Methuen, London, 1923 and Dover, New York, 1958.

CENTRIFUGAL FORCE

We shall now consider a particular kind of inertial force that always appears if the motion of a particle is described and analyzed from the standpoint of a *rotating* reference frame. This force—the centrifugal force—is familiar to us as the force with which, for example, an object appears to pull on us if we whirl it around at the end of a string.[1] To introduce it, we shall consider a situation of just this kind.

Suppose that a "tether ball" is being whirled around in horizontal circular motion with constant speed (Fig. 12–11). We shall analyze the motion of the ball as seen from two viewpoints: a stationary frame S, and a rotating frame S' that rotates with the same (constant) rotational speed as the ball. For convenience, we align the coordinate systems with their z and z' axes (as well as origins) coincident. The rotational speed of S' relative to S will be designated ω (in rad/sec). Figure 12–11 shows the analysis with respect to these two frames. The essential conclusions are these:

1. From the standpoint of the stationary (inertial) frame, the ball has an acceleration $(-\omega^2 r)$ toward the axis of rotation. The force, F_r, to cause this acceleration is supplied by the tethering cord, and we must have

$$(\text{In } S) \qquad F_r = -m\omega^2 r$$

2. From the standpoint of a frame that rotates so as to keep exact pace with the ball, the acceleration of the ball is zero. We can maintain the validity of Newton's law in the rotating frame if, in addition to the force F_r, the ball experiences an inertial force F_i, equal and opposite to F_r, and so directed radially outward:

$$(\text{In } S') \quad \begin{cases} F'_r = F_r + F_i = 0 \\ F_i = m\omega^2 r \end{cases}$$

The force F_i is then what we call the centrifugal force.

The magnitude of the centrifugal force can be established experimentally by an observer in the rotating frame S'. Let him hold a mass m stationary (as seen in his rotating frame) by

[1]The name "centrifugal" comes from the Latin: *centrum*, the center, and *fugere*, to flee.

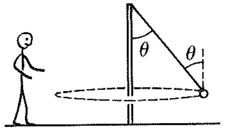
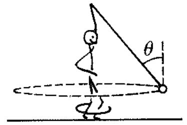
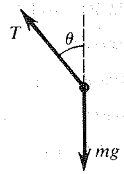
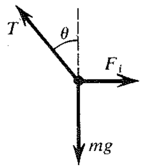
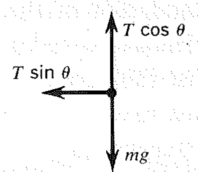
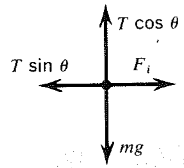
Procedure	Viewed from stationary frame S	Viewed from rotating frame S'
Pictorial sketch of problem	 Ball is observed to move with speed v in a circle of radius r (angular speed ω)	 Boy turns around at the same angular speed ω as the ball; from his point of view, the ball is at rest
"Isolate the body." Draw all forces that act on the ball		 T = tension in cord mg = force of gravity F_i = inertial force due to viewing the problem from a rotating frame
For ease of calculation we resolve forces into components in mutually perpendicular directions		
We now analyze the problem in terms of $\mathbf{F} = m\mathbf{a}$	<u>Vertical</u> Direction Because there is no vertical acceleration, we conclude that the net vertical force must be zero; hence $T \cos\theta = mg$ <u>Horizontal</u> Direction The object is traveling in a circle, therefore accelerating; the net force (i.e., the sum of all three forces) is horizontal toward the center of the circle, and must be equal in magnitude to mv^2/r; hence $T \sin\theta = \frac{mv^2}{r} = m\omega^2 r$ This force is directed radially <u>inward</u>	<u>Vertical</u> Direction Because there is no vertical acceleration, we conclude that the net vertical force must be zero; hence: $T \cos\theta = mg$ <u>Horizontal</u> Direction The object is "at rest," therefore the sum of all the forces on it must be zero; hence F_i is equal in magnitude to $T \sin\theta$. From the analysis in the left column, it is given by $F_i = \frac{mv^2}{r} = m\omega^2 r$ and is directed outward. We call F_i the <u>centrifugal</u> force

Fig. 12–11 Motion of a suspended ball, which is traveling in a horizontal circle, as analyzed from the earth's reference frame and from a frame rotating with the ball.

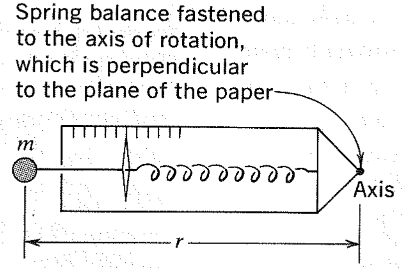


Fig. 12–12 Measurement of the force needed to hold an object at rest in a rotating reference frame.

attaching it to a spring balance (Fig. 12–12). If the mass is at any location except on the axis of rotation, the spring balance will show that it is exerting on the mass an inward force proportional to m and r. If the observer in S' is informed that his frame is rotating at the rate of ω rad/sec, he can confirm that this force is equal to $m\omega^2 r$. The observer explains the extension of the spring by saying that it is counteracting the outward centrifugal force on m which is present in the rotating frame. Furthermore, if the spring breaks, then the net force on the mass is just the centrifugal force and the object will at that instant have an outward acceleration of $\omega^2 r$ in response to this so-called "fictitious" force. Once again the inertial force is "there" by every criterion we can apply (except our inability to find another physical system as its source).

The magnitude of the centrifugal force is given, as we have seen, by the equation

$$F_{\text{centrifugal}} = m\frac{v^2}{r} = m\omega^2 r \qquad \text{(radially outward)} \qquad (12\text{–}6)$$

A nice example of our almost intuitive use of this force, under conditions in which there is nothing to balance it, is provided by situations such as the following: We have been washing a piece of straight tubing, and we want to get it dry on the inside. As a first step we get rid of the larger drops of water that are sitting on the inside walls. And we do this, not by shaking the tube longitudinally, but by whirling it in a circular arc [Fig. 12–13(a)]. The analysis of what happens as we begin this rotation gives a particularly clear picture of the difference between the descriptions of the process in stationary and rotating frames. It also provides us with a different way of deriving the formula for the centrifugal force itself.

Suppose that a drop, of mass m, is sitting on the inner wall of the tube at a point A [Fig. 12–13(b)], a distance r from the axis of rotation. Assume that the tube is very smooth, so that

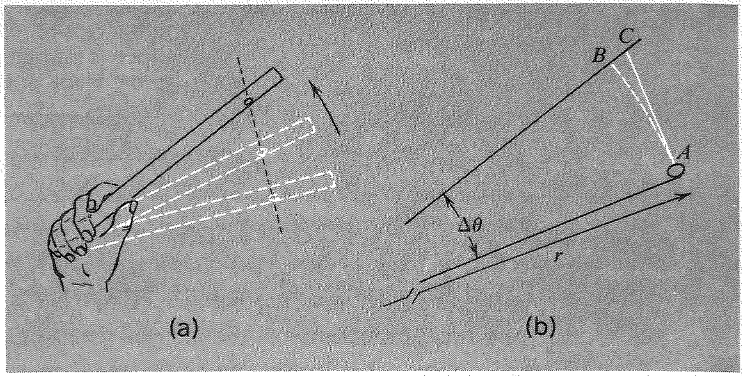


Fig. 12–13 (a) Shaking a drop of water out of a tube. (b) Analysis of initial motion in terms of centrifugal forces.

the drop encounters no resistance if it moves along the tube. The drop must, however, be carried along in any transverse movement of the tube resulting from the rotation. Then if the tube is suddenly set into motion and rotated through a small angle $\Delta\theta$, the drop, receiving an impulse normal to the wall of the tube at A, moves along the straight line AC. This, however, means that it is now further from the axis of rotation than if it had been fixed to the tube and had traveled along the circular arc AB. We have, in fact,

$$BC = r \sec \Delta\theta - r$$

Now

$$\sec \Delta\theta = (\cos \Delta\theta)^{-1} \approx [1 - \tfrac{1}{2}(\Delta\theta)^2]^{-1} \approx 1 + \tfrac{1}{2}(\Delta\theta)^2$$

Therefore,

$$BC \approx \tfrac{1}{2}r(\Delta\theta)^2$$

We can, however, express $\Delta\theta$ in terms of the angular velocity ω and the time Δt: $\Delta\theta = \omega\,\Delta t$. Thus we have

$$BC = \tfrac{1}{2}\omega^2 r(\Delta t)^2$$

This is then recognizable as the radial displacement that occurs in time Δt under an acceleration $\omega^2 r$. Hence we can put

$$a_{\text{centrifugal}} = \omega^2 r$$

and so

$$F_{\text{centrifugal}} = m\omega^2 r$$

Notice, then, that what is, in fact, a small transverse displacement in a straight line, with no real force in the radial direction, appears in the frame of the tube as a small, purely radial displacement under an unbalanced centrifugal force. The physical fact that the drop is moved outward along the tube is readily understood in terms of either description. (We should add, however, that our analysis as it stands does only apply to the initial step of the motion. Once the drop has acquired an appreciable radial velocity, things become more complicated.)

The term "centrifugal force" is frequently used incorrectly. For example, one may read such statements as "The satellite does not fall down as it moves around the earth because the centrifugal force just counteracts the force of gravity and hence there is no net force to make it fall." Any such statement flouts Newton's first law—A body with no net force on it travels in a *straight* line. . . . For if the satellite is described as *moving* in a curved path around the earth, it must also have an unbalanced force on it. The only frame in which the centrifugal force does balance the gravitational force is the frame in which the satellite appears not to move at all. One can, of course, consider the description of such motions with respect to a reference frame rotating at some arbitrary rate different from that of the orbiting object itself. In this case, however, the centrifugal contribution to the inertial forces represents only a part of the story, and the simple balancing of "real" and centrifugal forces does not apply. In particular, let us reemphasize that in a nonrotating frame of reference there is no such thing as centrifugal force. The long-standing confusion that leads people to use the term "centrifugal force" incorrectly has driven at least one author to extreme vexation. In an otherwise sober and quite formal text the author writes: "There is no answer to these people. Some of them are good citizens. They vote the ticket of the party that is responsible for the prosperity of the country; they belong to the only true church; they subscribe to the Red Cross drive—but they have no place in the Temple of Science; they profane it."[1]

CENTRIFUGES

The laboratory centrifuge represents an immensely important and direct application of the dynamical principle of centrifugal force. The basic arrangement of a simple type of centrifuge is

[1]W. F. Osgood, *Mechanics*, Macmillan, New York, 1937.

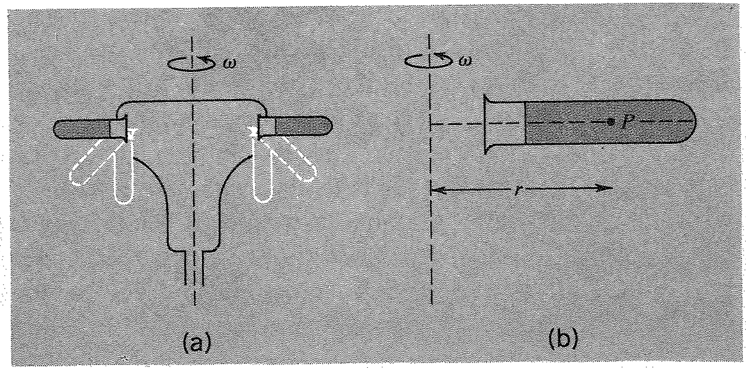


Fig. 12–14 (a) Vertical section through a simple centrifuge. (b) Analysis of radial sedimentation in terms of centrifugal forces.

shown in Fig. 12–14(a). Carefully balanced tubes of liquid are suspended on smooth pivots from a rotor. When the rotor is made to spin at high speed, the tubes swing upward and outward into almost horizontal positions and may be maintained in this orientation for many hours on end. At any point P in one of the tubes [Fig. 12–14(b)], distance r from the axis of rotation, there is an effective gravitational field of magnitude $\omega^2 r$, which may be made very much greater than g. For example, if r = 15 cm and the rotor spins at 25 rps ($\omega = 50\pi$ sec^{-1}), the value of $\omega^2 r$ is about 4000 m/sec^2 or 400 g. Small particles in suspension in the liquid will be driven toward the outward (bottom) end of the tube much more quickly than they would ever be under the action of gravity alone.

The basis for calculating the drift speed is the formula for the resistive force to motion through a fluid at low speeds, which we first met in Chapter 5. For a spherical particle of radius r and speed v, this force is proportional to the product rv. If the medium is water, the approximate magnitude of the force is given by

$$R(v) \approx 0.02rv$$

where R is in newtons, r in meters, and v in m/sec. A steady value of v is attained when this force just balances the driving force associated with the effective gravitational field strength, g'. In calculating this driving force it is important to allow for buoyancy effects—i.e., Archimedes' principle. If the density of the particle is ρ_p and the density of the liquid is ρ_l, the driving force is given by

$$F = \frac{4\pi}{3}(\rho_p - \rho_l)r^3 g'$$

This can be more simply expressed if we introduce the true mass m of the particle ($= 4\pi\rho_p r^3/3$), in which case we can put

$$F = \left(1 - \frac{\rho_l}{\rho_p}\right) mg'$$

To take a specific example, suppose that we have an aqueous suspension of bacterial particles of radius 1 μ, each with a mass of about 5×10^{-15} kg and a density about 1.1 times that of water. If we take for g' the value 400 g calculated earlier, we find

$$F \approx 2 \times 10^{-12}\ \text{N}$$

We thus obtain a drift speed given by

$$v \approx \frac{2 \times 10^{-12}}{2 \times 10^{-2} \times 10^{-6}} \approx 10^{-4}\ \text{m/sec}$$

This represents a settling rate of several centimeters per hour, which makes for effective separation in reasonable times, whereas under the normal gravity force alone one would have only a millimeter or two per day.

The above example represents what one may regard as a more or less routine type of centrifugation, but in 1925 the Swedish chemist T. Svedberg opened up a whole new field of research when, by achieving centrifugal fields thousands of times stronger than g, he succeeded in measuring the molecular weights of proteins by studying their radial sedimentation. The type of machine he developed for this purpose was appropriately named the *ultracentrifuge*, and Svedberg succeeded in producing centrifugal fields as high as about 50,000 g. The physicist J. W. Beams has taken the technique even further through his development of magnetic suspensions, in vacuum, that dispense with mechanical bearings altogether. The rotor simply spins in empty space, with carefully controlled magnetic fields to hold it at a constant vertical level against the normal pull of gravity. By such methods Beams has produced centrifugal fields equivalent to about 10^6 g in a usable centrifuge and fields as high as 10^9 g in tiny objects (e.g., spheres of 0.001-in. diameter). The limitation is set by the bursting speed of the rotor; this defines a maximum value of ω proportional to $1/r$ (see Chapter 7, p. 208). Since the centrifugal field g' is equal to $\omega^2 r$, and the limiting speed sets an upper limit to ωr, it may be seen that the attainable value of g' varies as $1/r$.

The technique of ultracentrifuge methods has been brought to an extraordinary pitch of refinement. It has become possible to determine molecular weights to a precision of better than 1% over a range from about 10^8 (virus particles) down to as low as about 50. The possibility of measuring the very low molecular weights by this method is particularly impressive. Beams has pointed out that in a solution of sucrose in water, the calculated rate of descent of an individual sucrose molecule, of mass about 340 amu and radius about 5 Å, would (according to the kind of analysis we gave earlier) be less than 1 mm in 100 years under normal gravity. (A rate as slow as this becomes in fact meaningless because, as Beams points out, it would be completely swamped by random thermal motions.) If a field of $10^5\,g$ is available, however, the time constant of the sedimentation process is reduced to the order of 1 day or less, which brings the measurement well within the range of possibility.[1]

This whole subject of centrifuges and centrifugation is a particularly good application of the concept of inertial force, because the phenomena are so appropriately described in terms of static or quasistatic equilibrium in the rotating frame.

CORIOLIS FORCES

We have seen how the centrifugal force, $m\omega^2 r$, exerted on a particle of a given mass m in a frame rotating at a given angular velocity ω, depends only on the distance r of the particle from the axis of rotation. In general, however, another inertial force appears in a rotating frame. This is the *Coriolis force*,[2] and it depends only on the velocity of the particle (not on its position). We shall introduce this force in a simple way for some specific situations. Later, by introducing *vector* expressions for rotational motion, we shall develop a succinct notation that gives both the centrifugal and Coriolis forces in a form valid in three dimensions using any type of coordinate system.

The need to introduce the Coriolis force is easily shown by comparing the straight-line motion of a particle in an inertial frame S with the motion of the same particle as seen in a rotating frame S'.

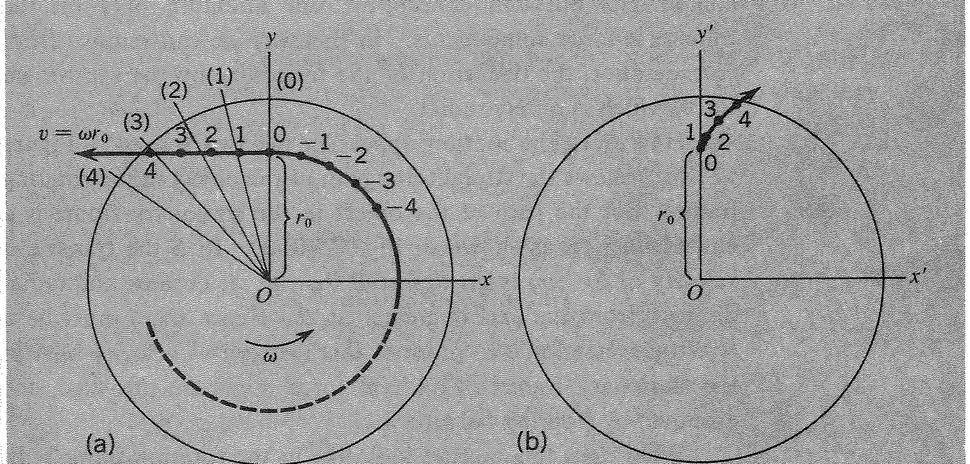
[1]For further reading on this extremely interesting subject, see T. Svedberg and K. O. Pedersen, *The Ultracentrifuge*, Oxford University Press, New York, 1960, and J. W. Beams, "High Centrifugal Fields," *Physics Teacher* **1**, 103 (1963).
[2]G. Coriolis, *J. de l'Ecole Polytechnique, Cahier* **24**, 142 (1835).

Suppose that S' is a coordinate system attached to a horizontal circular table that rotates with constant angular speed ω. Let the vertical axis of rotation define the z' axis and suppose that the table surface (in the $x'y'$ plane) has no friction. A string fastened to the origin holds a particle on the y' coordinate axis at a radial distance r_0' from the axis of rotation. Thus, in the S' frame, the particle is at rest in equilibrium under the combined forces of the tension in the string and the centrifugal force. (The vertical force of gravity and the normal force of the table surface always add to zero and need not concern us further.)

The same particle is viewed from an inertial frame S which coincides with S' at $t = 0$. In this stationary frame, the particle travels with uniform speed $v_\theta = \omega r_0$ in a circle of constant radius r_0 $(= r_0')$ under the single unbalanced force of the tension in the string. There is, of course, no centrifugal force in this inertial frame.

At $t = 0$ the string breaks. In S the particle then travels

Fig. 12–15 Two different descriptions of the motion of an object that is initially tethered on a rotating disk and begins motion under no forces at $t = 0$.



As seen in the stationary S frame

At $t = 0$, the string breaks. The dots indicate the positions of the particle at successive equal time intervals 0, 1, 2, 3,.... At the same time instants, the rotating y' axis has the positions shown by (0), (1), (2), (3),....

As seen in the rotating S' frame

The particle is initially at rest. After the string breaks, it begins to move radially outward, but as soon as it acquires some speed, it veers toward the right of the y' axis.

in a straight line with constant speed $v = \omega r_0$ as shown in Fig. 12–15(a). To find the motion in S', we compare *in the stationary frame* the positions of the y' axis and the corresponding locations of the particle at successive equally-spaced times. We discover that the particle, as observed in S', not only moves radially outward, but also moves farther and farther to the right of the single radial line formed by the rotating y' axis. This result is plotted in Fig. 12–15(b). To explain this motion as observed in the rotating frame, it is necessary to postulate, in addition to the centrifugal force, a sideways deflecting force. This deflecting force is the *Coriolis* force. In the course of the following discussion, we shall determine its magnitude and show that it always acts at right angles to any velocity $\mathbf{v}'$ in the S' frame.

We can find the magnitude of the Coriolis force by investigating another simple motion in these two frames. Suppose that, instead of the situation just described, we make a particle follow a radially outward path in the rotating frame at *constant* velocity $\mathbf{v}'_r$. In this frame there must be no net force on the particle. Hence we shall have to supply some real (inward) force to counteract the varying (outward) centrifugal force as the particle moves. We shall not concern ourselves with these radial components but will concentrate our attention only on the transverse-force components. In this way we can remove from consideration the distortion of the trajectory by the centrifugal force, which is purely radial.

How does the motion appear in the two frames? Figure 12–16(b) shows the straight-line path of the object in the rotating frame. But the path of the object in the stationary frame is a curved line AB as shown in Fig. 12–16(a). In S the transverse velocity v_θ $(= \omega r)$ is greater at B than at A, because the radial distance from the axis is greater at B. Hence there must be a *real* transverse force to produce this increase of velocity seen in the stationary frame. This real force might be provided, for example, by a spring balance.

What does this motion look like in the rotating frame? In S' the object moves outward with constant speed and hence has no acceleration [see Fig. 12–16(b)]. This means, as we have said, that there can be no net force on the object in the rotating frame. But since an observer in S' sees the spring balance exerting a *real* sideways force on the object in the $+\theta$ direction, he infers that there is a counteracting *inertial* force in the $-\theta$ direction to balance it. This is the Coriolis force.

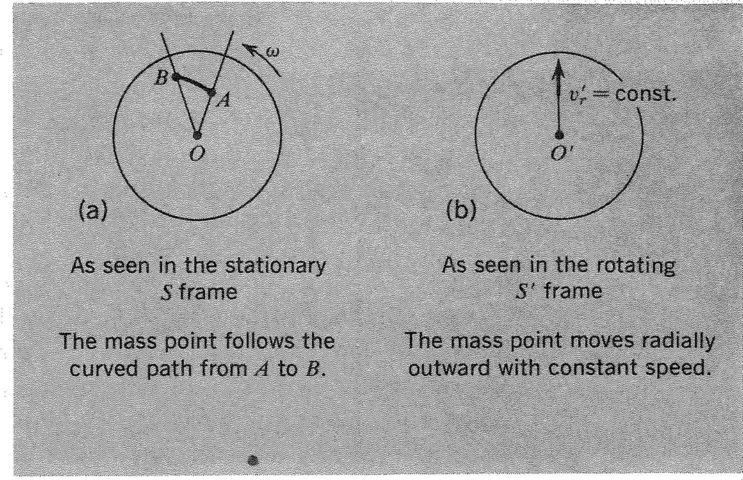


Fig. 12–16 (a) Laboratory view of the path of a particle that moves radially outward on a rotating table. (b) The motion as it appears in the rotating frame itself.

To determine its magnitude, let OA and OB in Fig. 12–17 be successive positions of the same radial line at times separated by Δt. Let OC be the bisector of the angle $\Delta\theta$. The velocity perpendicular to OC changes by the amount Δv_θ during Δt, where

$$\Delta v_\theta = [\omega(r + \Delta r)\cos(\Delta\theta/2) + v_r \sin(\Delta\theta/2)] - [\omega r \cos(\Delta\theta/2) - v_r \sin(\Delta\theta/2)]$$

For small angles we can put the cosine equal to 1 and the sine equal to the angle, which leads to the following very simple expression for Δv_θ:

$$\Delta v_\theta \approx \omega\,\Delta r + v_r\,\Delta\theta$$

The transverse acceleration a_θ is thus given by

$$a_\theta = \omega(\Delta r/\Delta t) + v_r(\Delta\theta/\Delta t)$$

But

$$\Delta r/\Delta t = v_r \qquad \text{and} \qquad \Delta\theta/\Delta t = \omega$$

Hence

$$a_\theta = 2\omega v_r$$
$$F_\theta = 2m\omega v_r$$

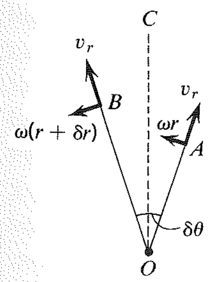


Fig. 12–17 Basis of calculating the Coriolis force for a particle moving radially at constant speed with respect to a rotating table.

This gives us the *real* force needed to cause the *real* acceleration as judged in S. But as observed in the rotating frame S', there is no acceleration and no net force. Hence the existence of the Coriolis force, equal to $-2m\omega v'_r$, is inferred. (Note that $v_r = v'_r$.) This inertial force is in the negative θ' direction, opposite to the spring force, and is at right angles to the direction of motion of the particle:

$$F'_\theta(\text{Coriolis}) = -2m\omega v'_r \qquad (12\text{–}7)$$

An important feature, which you should verify for yourself, is that if we had considered a radially *inward* motion (v'_r negative), then we would have inferred the existence of a Coriolis force acting in the *positive* θ' direction. In both cases, therefore, the Coriolis force acts to deflect the object in the same way with respect to the direction of the velocity $\mathbf{v}'$ itself—to the right if the frame S' is rotating counterclockwise, as we have assumed, or to the left if S' rotates clockwise. It turns out, in fact, as we shall prove later, that even in the case of motion in an arbitrary direction the Coriolis force is always a *deflecting* force, exerted at right angles to the direction of motion as observed in the rotating frame.

The Coriolis force is very real from the viewpoint of the rotating frame of reference. If you want to convince yourself of the reality of this "fictitious" force, ride a rotating merry-go-round and try walking a radial line outward or inward. (Proceed cautiously—the Coriolis force is so unexpected and surprising that it is easy to lose one's balance!)

DYNAMICS ON A MERRY-GO-ROUND

As we have just mentioned, the behavior of objects in motion within a rotating reference frame can run strongly counter to one's intuitions. It is not too hard to get used to the existence of the centrifugal force acting on an object at rest with respect to the rotating frame, but the combination of centrifugal and Coriolis effects that appear when the object is set in motion can be quite bewildering, and sometimes entertaining. Suppose, for example, that a man stands at point A on a merry-go-round [Fig. 12–18(a)] and tries to throw a ball to someone at B (or perhaps aims for the bull's-eye of a dart board placed there). Then the thrown object mysteriously veers to the right and misses its target every time. One can blame part of this, of course, on the centrifugal force itself. However, it is to be noted that

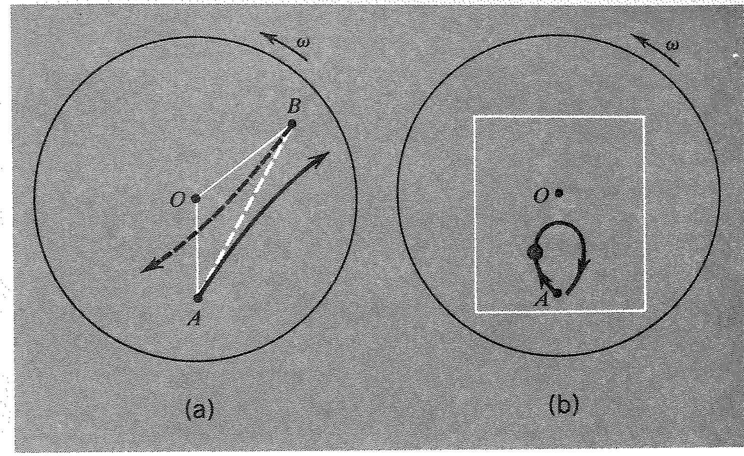


Fig. 12–18 (a) Trajectories of objects as they appear to observers on a rotating table. (b) An object projected on a frictionless rotating table can return to its starting point.

since the magnitude of the centrifugal force is $m\omega^2 r$ and that of the Coriolis force is $2m\omega v'$, the ratio of these two forces is proportional to $v'/\omega r$. Thus if v' is made much greater than the actual peripheral speed of the merry-go-round, the peculiarities of the motion are governed almost entirely by the Coriolis effects. If this condition holds, the net deflection of a moving object will always be to the right with respect to $\mathbf{v}'$ on a merry-go-round rotating counterclockwise. Thus if the positions A and B in Fig. 12–18(a) are occupied by two people trying to throw a ball back and forth, each will have to aim to the left in order to make a good throw.

An extreme case of this kind of behavior can cause an object to follow a continuously curved path that brings it back to its starting point, although it is not subjected to any real forces at all. This phenomenon has been demonstrated in the highly entertaining and instructive film, *Frames of Reference.*[1] A dry-ice puck, launched at point A on a tabletop of plate glass [Fig. 12–18(b)], can be caused by a skilled operator to follow a trajectory of the kind indicated.

GENERAL EQUATION OF MOTION IN A ROTATING FRAME[2]

The goal of this discussion will be to relate the time derivatives of the displacement of a moving object as observed in a sta-

[1] "*Frames of Reference*," by J. N. P. Hume and D. G. Ivey, Education Development Center, Newton, Mass., 1960.

[2] This section may be omitted by a reader who is willing to take on trust its final results—that the total inertial force in a rotating frame is the combination of the centrifugal force with a Coriolis force corresponding to a generalized form of Eq. (12–7).

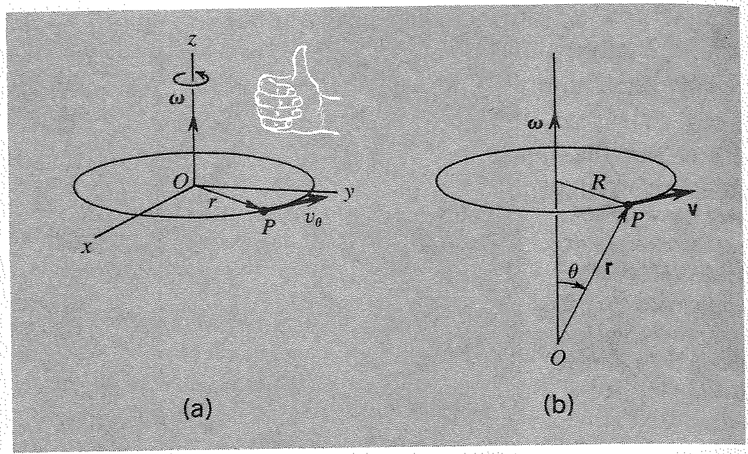


Fig. 12–19 Use of angular velocity as a vector to define the linear velocity of a particle on a rotating table: $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$.

tionary frame S and in a rotating frame S'. To set the stage, we shall introduce the idea that angular velocity may be represented as a vector.

Consider first a point P on a rotating disk [Fig. 12–19(a)]. It has a purely tangential velocity, v_θ, in a direction at right angles to the radius OP. We can describe this velocity, in both magnitude and direction, if we define a *vector* according to the same convention that we introduced for torque in Chapter 4. That is, if the fingers of the *right* hand are curled around in the sense of rotation, keeping the thumb extended as shown in the figure, then $\boldsymbol{\omega}$ is represented as a vector, of length proportional to the angular speed, in the direction in which the thumb points. Thus with $\boldsymbol{\omega}$ pointing along the positive z direction, one is defining a rotation that carries each point such as P from the positive x direction toward the positive y direction. The rotation of the disk is in this case counterclockwise as seen from above.

The velocity of P is now given by the vector (cross) product of $\boldsymbol{\omega}$ with the radius vector $\mathbf{r}$:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \qquad (12\text{–}8)$$

This vector-product expression is valid in three dimensions also, if the position vector $\mathbf{r}$ of P is measured from any point on the axis of rotation, as shown in Fig. 12–19(b). The radius of the circle in which P moves is $R = r \sin\theta$. Thus we have $v = v_\theta = \omega r \sin\theta$, in a direction perpendicular to the plane defined by $\boldsymbol{\omega}$ and $\mathbf{r}$. That is precisely what Eq. (12–8) gives us.

Next, we consider how the change of *any* vector during a small time interval Δt can be expressed as the vector sum of two contributions:

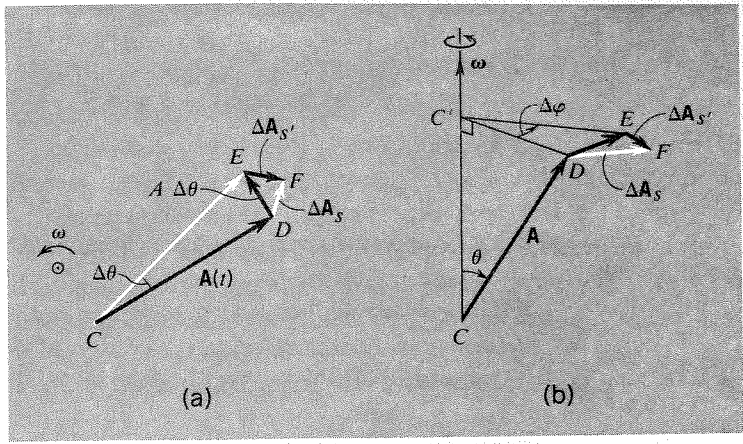


Fig. 12–20 (a) Change of a vector, analyzed in terms of its change as measured on a rotating table, together with the change due to rotation of the table itself. (b) Similar analysis for an arbitrary vector referred to any origin on the axis of rotation.

1. The change that would occur if it were simply a vector of constant length embedded in the rotating frame S'.
2. The further change described by its change of length and direction as observed in S'.

In Fig. 12–20(a) we show this analysis for motion confined to a plane. The vector $\mathbf{A}$ at time t is represented by the line CD. If it remains fixed with respect to a rotating table, its direction at time $t + \Delta t$ is given by the line CE, where $\Delta\theta = \omega\,\Delta t$. Thus its change due to the rotation alone would be represented by DE, where $DE = A\,\Delta\theta = A\omega\,\Delta t$. From the standpoint of frame S' this change would not be observed. There might, however, be a change represented by the line EF; we shall denote this as $\Delta\mathbf{A}_{S'}$—the change of $\mathbf{A}$ as observed in S'. The vector sum of DE and EF, i.e., the line DF, then represents the true change of $\mathbf{A}$ as observed in S. We therefore denote this as $\Delta\mathbf{A}_S$.

In Fig. 12–20(b) we show the corresponding analysis for three dimensions. The length of DE is now equal to $A \sin\theta\,\Delta\varphi$; its direction is perpendicular to the plane defined by $\boldsymbol{\omega}$ and $\mathbf{A}$. Since $\Delta\varphi = \omega\,\Delta t$, we can put

$$\text{vector displacement } DE = (\boldsymbol{\omega} \times \mathbf{A})\,\Delta t$$

The displacement $\Delta\mathbf{A}_{S'}$ may be in any direction with respect to DE, but the two again combine to give a net displacement DF which is to be identified with $\Delta\mathbf{A}_S$. Thus we have

$$\Delta \mathbf{A}_S = \Delta \mathbf{A}_{S'} + (\boldsymbol{\omega} \times \mathbf{A})\,\Delta t$$

We can at once proceed from this to a relation between the rates of change of **A** as observed in S and S', respectively:

$$\left(\frac{d\mathbf{A}}{dt}\right)_S = \left(\frac{d\mathbf{A}}{dt}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{A} \qquad (12\text{–}9)$$

This is a very powerful relation because **A** can be any vector we please.

First, we shall choose **A** to be the position vector **r**. Then $(d\mathbf{A}/dt)_S$ is the true velocity, **v**, as observed in S, and $(d\mathbf{A}/dt)_{S'}$ is the apparent velocity, **v**′, as observed in S'. Thus we immediately have

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r} \qquad (12\text{–}10)$$

Next, we shall choose **A** to be the velocity **v**:

$$\left(\frac{d\mathbf{v}}{dt}\right)_S = \left(\frac{d\mathbf{v}}{dt}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{v} \qquad (12\text{–}11)$$

Now $(d\mathbf{v}/dt)_S$ is the true acceleration, **a**, as observed in S. The quantity $(d\mathbf{v}/dt)_{S'}$ is, however, a sort of hybrid—it is the rate of change in S' of the velocity as observed in S. We can make more sense of this if we substitute for **v** from Eq. (12–10); we then have

$$\left(\frac{d\mathbf{v}}{dt}\right)_{S'} = \left(\frac{d\mathbf{v}'}{dt}\right)_{S'} + \boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{S'}$$

The two terms on the right of this equation are now quite recognizable; $(d\mathbf{v}'/dt)_{S'}$ is the acceleration, **a**′, as observed in S', and $(d\mathbf{r}/dt)_{S'}$ is just **v**′. Thus we have

$$\left(\frac{d\mathbf{v}}{dt}\right)_{S'} = \mathbf{a}' + \boldsymbol{\omega} \times \mathbf{v}'$$

Substituting this in Eq. (12–11) we thus get

$$\mathbf{a} = \mathbf{a}' + \boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{v}$$

We do not need to have both **v** and **v**′ on the right-hand side, and we shall again substitute for **v** from Eq. (12–10). This gives us finally

$$\mathbf{a} = \mathbf{a}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad (12\text{–}12)$$

A remark is in order regarding the last term, which involves the cross product of three vectors. According to the rules of vector algebra, the cross product inside the parentheses is to be taken first, then the other cross product performed. A nonzero answer will result for all cases where the angle formed by $\boldsymbol{\omega}$ and **r** is other than 0° or 180°. Performing the cross products in the reverse (incorrect) order, however, would result in zero for all cases, regardless of the angle between these vectors.

Multiplying Eq. (12–12) throughout by the mass m of the object, we recognize the left side as the net external force on the mass as seen in the stationary system.

$$m\mathbf{a} = \mathbf{F}_{\text{net}} = m\mathbf{a}' + 2m(\boldsymbol{\omega} \times \mathbf{v}') + m[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})]$$

In the rotating frame of reference, the object m has the acceleration **a**′. We may preserve the format of Newton's second law in this accelerated frame of reference by rearranging the above equation, so as to be able to write

$$\mathbf{F}'_{\text{net}} = m\mathbf{a}' \qquad (12\text{–}13a)$$

where

$$\mathbf{F}'_{\text{net}} = \underset{\text{"real" force}}{\mathbf{F}_{\text{net}}} \underbrace{- \underset{\text{Coriolis force}}{2m(\boldsymbol{\omega} \times \mathbf{v}')} - \underset{\text{centrifugal force}}{m[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})]}}_{\text{inertial forces}} \qquad (12\text{–}13b)$$

The mathematical form of Eq. (12–13b) shows that both the Coriolis force and the centrifugal force are in a direction at right angles to the axis of rotation defined by $\boldsymbol{\omega}$. The centrifugal force, in particular, is always radially outward from the axis, as is clear if one considers the geometrical relationships of the vectors involved in the product $-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$, as shown in Fig. 12–21. The equation also shows that the Coriolis force

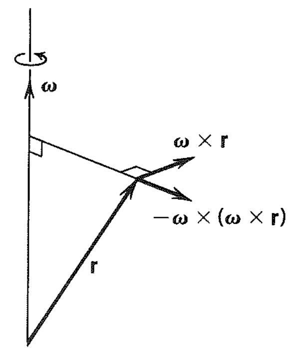


Fig. 12–21 Relation of the vectors involved in forming the centrifugal acceleration $-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$.

would reverse if the direction of ω were reversed, but the direction of the centrifugal force would remain unchanged.

The specification of $\mathbf{F}'$ in Eq. (12–13) can be made entirely on the basis of measurements of position, velocity, and acceleration as observed within the rotating frame itself. The centrifugal term, involving the vector $\mathbf{r}$, might seem to contradict this, but we could just as well put $\mathbf{r}'$ instead of $\mathbf{r}$, because observers in the two frames do agree on the vector position of a moving object at a given instant, granted that they use the same choice of origin.

To summarize, we have established by the above calculation that the dynamics of motion as observed in a uniformly rotating frame of reference may be analyzed in terms of the following three categories of forces:

"Real": $\mathbf{F}_{net}$	This is the sum of all the "real" forces on the object such as forces of contact, tensions in strings, the force of gravity, electrical forces, magnetic forces, and so on. Only these forces are seen in a stationary frame of reference.
Coriolis: $-2m(\boldsymbol{\omega} \times \mathbf{v}')$	The Coriolis force is a *deflecting* force always at right angles to the velocity $\mathbf{v}'$ of the mass m. If the object has no velocity in the rotating frame of reference, there is no Coriolis force. It is an inertial force *not* seen in a stationary frame of reference. Note minus sign.
Centrifugal: $-m[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})]$	The centrifugal force depends on position only and is always radially outward. It is an inertial force *not* seen in a stationary frame of reference. We could equally well write it as $-m[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')]$. Note the minus sign.

THE EARTH AS A ROTATING REFERENCE FRAME

In this section we shall consider a few examples of the way in which the earth's rotation affects the dynamical processes occurring on it.

The local value of g

If a particle P is at rest at latitude λ near the earth's surface, then as judged in the earth's frame it is subjected to the gravitational force $\mathbf{F}_g$ and the centrifugal force $\mathbf{F}_{cent}$ shown in Fig. 12–22(a). The magnitude of the latter is given, according to Eq. (12–13b), by the equation

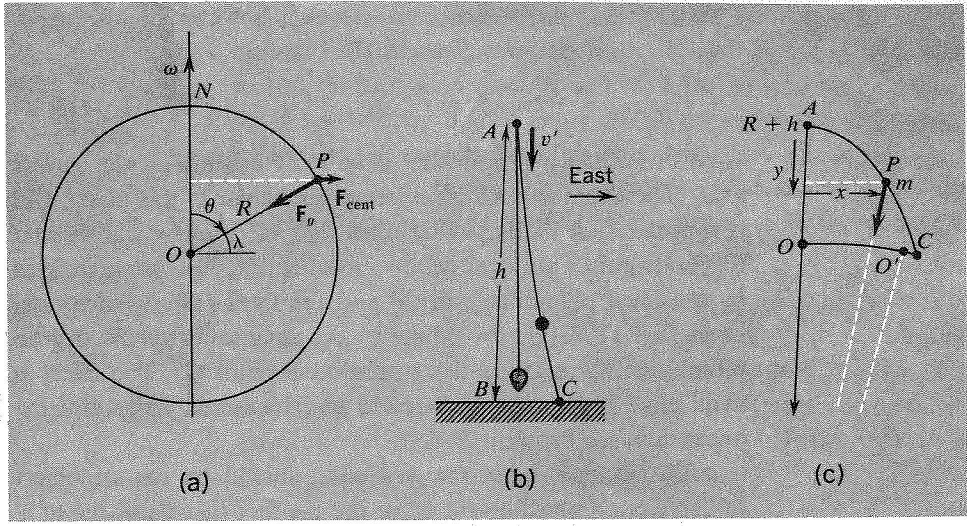


Fig. 12–22 (a) Forces on an object at rest on the earth, as interpreted in a reference frame that rotates with the earth. (b) An object falling from rest relative to the earth undergoes an eastward displacement. (c) The falling motion of (b), as seen from a frame that does not rotate with the earth.

$$F_{cent} = m\omega^2 R \sin\theta = m\omega^2 R \cos\lambda$$

where R is the earth's radius. We have already discussed in Chapter 8 the way in which this centrifugal term reduces the local magnitude of g and also modifies the local direction of the vertical as defined by a plumbline. The analysis is in fact much simpler and clearer from the standpoint of our natural reference frame as defined by the earth itself. We have, as Fig. 12–22(a) shows, the following relations:

$$\begin{aligned} F'_r &= F_g - F_{cent}\cos\lambda = F_g - m\omega^2 R\cos^2\lambda \\ F'_\theta &= F_{cent}\sin\lambda = m\omega^2 R \sin\lambda\cos\lambda \end{aligned} \tag{12–14}$$

Deviation of freely falling objects

If a particle is released from rest at a point such as P in Fig. 12–22(a), it begins to accelerate downward under the action of a net force $\mathbf{F}'$ whose components are given by Eq. (12–14). As soon as it has any appreciable velocity, however, it also experiences a Coriolis force given by the equation

$$\mathbf{F}_{\text{Coriolis}} = -2m\boldsymbol{\omega} \times \mathbf{v}' \tag{12–15}$$

Now the velocity $\mathbf{v}'$ is in the plane PON containing the earth's axis. The Coriolis force must be perpendicular to this plane, and a consideration of the actual directions of $\boldsymbol{\omega}$ and $\mathbf{v}'$ shows that it is eastward. Thus if we set up a local coordinate system defined by the local plumbline vertical and the local easterly direction, as in Fig. 12–22(b), the falling object deviates eastward from a plumbline AB and hits the ground at a point C. The effect is very small but has been detected and measured in careful experiments (see Problem 12–24).

To calculate what the deflection should be for an object falling from a given height h, we use the fact that the value of v' to be inserted in Eq. (12–15) is extremely well approximated by the simple equation of free vertical fall:

$$v' = gt$$

where v' is measured as positive downward. Thus if we label the eastward direction as x', we have

$$m\frac{d^2x'}{dt^2} = (2m\omega\cos\lambda)gt$$

Integrating this twice with respect to t, we have

$$x' = \tfrac{1}{3}g\omega t^3\cos\lambda \tag{12–16}$$

For a total distance of vertical fall equal to h, we have $t = (2h/g)^{1/2}$, which thus gives us

$$x' = \frac{2\sqrt{2}}{3}\,\frac{\omega\cos\lambda}{g^{1/2}}\,h^{3/2} \tag{12–17}$$

Inserting approximate numerical values ($\omega = 2\pi\ \text{day}^{-1} \approx 7 \times 10^{-5}\ \text{sec}^{-1}$), one finds

$$x' \approx 2 \times 10^{-5}h^{3/2}\cos\lambda \qquad (x' \text{ and } h \text{ in m})$$

Thus, for example, with $h = 50$ m at latitude 45°, one has $x' \approx 5$ mm, or about $\frac{1}{4}$ in.

It is perhaps worth reminding oneself that the effects of inertial forces can always be calculated, if one wishes, from the standpoint of an inertial frame in which these forces simply do not exist. In the present case, one can begin by recognizing that a particle held at a distance h above the ground has a higher eastward velocity than a point on the ground below. For simplicity, let us consider how this operates at the equator ($\lambda = 0$). Figure 12–22(c) shows the trajectory of the falling object as seen in a nonrotating frame. The object has an initial horizontal velocity given by

$$v_{0x} = \omega(R + h)$$

After a time t it has traveled a horizontal distance x given, very nearly, by ωRt. With the object now at P (see the figure) the gravitational force acting on it has a very small component in the negative x direction. We have, in fact,

$$F_x \approx -\frac{x}{R}F_g \approx -mg\omega t$$

Hence

$$\frac{d^2x}{dt^2} \approx -g\omega t$$

Integrating once, we have

$$\frac{dx}{dt} \approx v_{0x} - \tfrac{1}{2}g\omega t^2$$

Substituting the value $v_{0x} = \omega(R + h)$, this gives, as a very good approximation,

$$\frac{dx}{dt} = \omega(R + h) - \tfrac{1}{2}g\omega t^2$$

Integrating a second time, we have

$$x = \omega(R + h)t - \tfrac{1}{6}g\omega t^3$$

However, the point O at the earth's surface is also moving, with a constant speed of ωR. Thus, when the falling object hits the ground at C, the point O has reached O', where $OO' = \omega Rt$. Hence we have

$$x' = O'C \approx \omega ht - \tfrac{1}{6}g\omega t^3$$

If we substitute $h = \frac{1}{2}gt^2$, we at once obtain the result given by

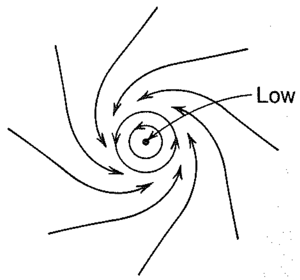


Fig. 12–23 Formation of a cyclone in the northern hemisphere, under the action of Coriolis forces on the moving air masses.

Eq. (12–16) for $\lambda = 0$. [Or, of course, we can substitute $t = \sqrt{2h/g}$ and arrive at Eq. (12–17)].

Patterns of atmospheric circulation

Because of the Coriolis effect, air masses being driven radially inward toward a low-pressure region, or outward away from a high-pressure region, are also subject to deflecting forces. This causes most cyclones to be in a counterclockwise direction in the northern hemisphere and clockwise in the southern hemisphere. The origin of these preferred rotational directions may be seen in Fig. 12–23, which shows the motions of air in the northern hemisphere moving toward a region of low pressure. The horizontal components of the Coriolis force deflect these motions

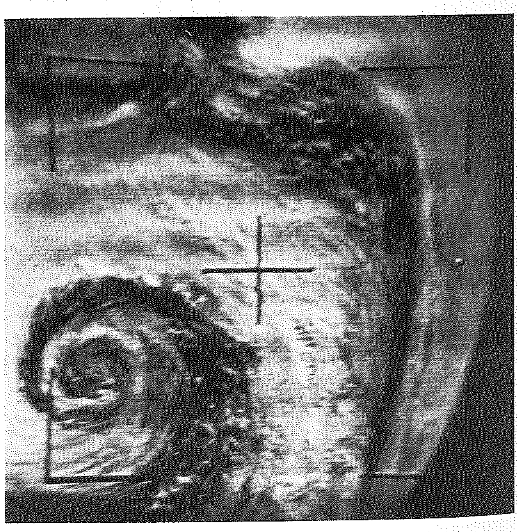


Fig. 12–24 Tiros satellite photograph of a cyclone. (Courtesy of Charles W. C. Rogers and N.A.S.A.)

toward the right. Thus, as the air masses converge on the center of the low-pressure region, they produce a net counterclockwise rotation. For air moving north or south over the earth's surface the Coriolis force is due east or due west, parallel to the earth's surface. If we consider a 1-kg mass of air at a wind velocity of 10 m/sec (about 22 mph) at 45° north latitude, a direct application of Eq. (12–15) gives us

$$F_{\text{Coriolis}} = 2m\omega v' \sin\lambda \approx (2)(1)(2\pi \times 10^{-5}) \times (10)(0.707) \approx 10^{-3}\ \text{N}$$

If we had considered air flowing in from east or west, the Coriolis forces would not be parallel to the earth's surface, but their components parallel to the surface would be given by the same equation as that used above. (Verify this.)

The approximate radius of curvature of the resultant motion may be obtained from

$$F = m\frac{v^2}{R}$$

or

$$R = m\frac{v^2}{F_{\text{Coriolis}}} \approx 1 \times \frac{100}{10^{-3}} = 10^5\ \text{m (about 60 miles)}$$

As air masses move over hundreds of miles on the earth's surface, they often form huge vortices—as is dramatically shown in the Tiros weather satellite photograph in Fig. 12–24.

Occasionally one reads that water draining out of a basin also circulates in a preferred direction because of the Coriolis force. In most cases, the Coriolis force on the flowing water is negligible compared with other larger forces which are present; however, if extremely precise and careful experiments are performed, the effect can be demonstrated.[1]

The Foucault pendulum

No account of Coriolis forces would be complete without some mention of the famous pendulum experiment named after the French physicist J. B. L. Foucault, who first demonstrated in 1851 how the slow rotation of the plane of vibration of a pen-

[1]See, for example, the film "Bathtub Vortex," an excerpt from "Vorticity," by A. H. Shapiro, National Council on Fluid Mechanics, 1962.

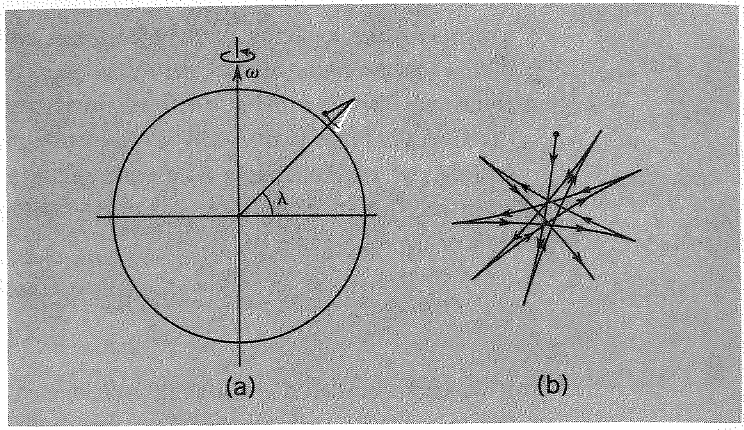


Fig. 12–25 (a) A pendulum swinging along a north–south line at latitude λ. (b) Path of pendulum bob, as seen from above. (The change of direction per swing is, however, grossly exaggerated.)

dulum could be used as evidence of the earth's own rotation.

It is easy, but rather too glib, to say that of course we are simply seeing the effect of the earth turning beneath the pendulum. This description might properly be used for a pendulum suspended at the north or south pole. One can even press things a little further and say that at a given latitude, λ [see Fig. 12–25(a)] the earth's angular velocity vector has a component $\omega \sin \lambda$ along the local vertical. This would indeed lead to the correct result—that the plane of the pendulum rotates at a rate corresponding to one complete rotation in a time $T(\lambda)$ given by

$$T(\lambda) = \frac{2\pi}{\omega \sin \lambda} = 24 \csc \lambda \quad \text{hours} \tag{12–18}$$

But the pendulum is, after all, connected to the earth via its suspending wire, and both the tension in the wire and the gravitational force on the bob lie in the vertical plane in which the pendulum is first set swinging. (So, too, is the air resistance, if this needs to be considered.) It is the Coriolis force that can be invoked to give a more explicit basis for the rotation. For a pendulum swinging in the northern hemisphere, the Coriolis force acts always to curve the path of the swinging bob to the right, as indicated in exaggerated form in Fig. 12–25(b). As with the Coriolis force on moving air, the effect does not depend on the direction of swing—contrary to the intuition most of us probably have that the rotation is likely to be more marked when the pendulum swings along a north–south line than when it swings east–west.

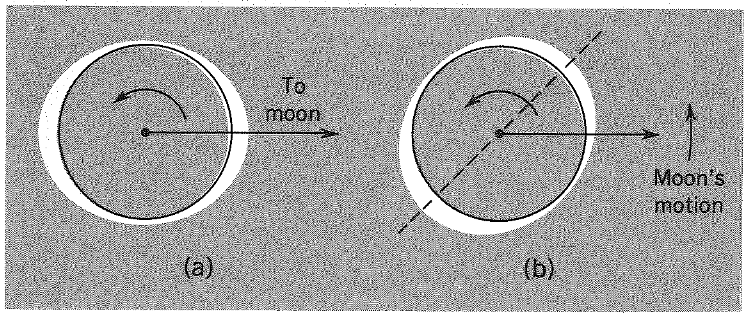
THE TIDES

As everyone knows, the production of ocean tides is basically the consequence of the gravitational action of the moon—and, to a lesser extent, the sun. Thus we could have discussed this as an example of universal gravitation in Chapter 8. The analysis of the phenomenon is, however, considerably helped by introducing the concept of inertial forces as developed in the present chapter.

The feature that probably causes the most puzzlement when one first learns about the tides is the fact that there are, at most places on the earth's surface, two high tides every day rather than just one. This corresponds to the fact that, at any instant, the general distribution of ocean levels around the earth has two bulges. On the simple model that we shall discuss, these bulges would be highest at the places on the earth's surface nearest to and farthest from the moon [Fig. 12–26(a)]. While the earth performs its rotation during 24 hr, the positions of the bulges would remain almost stationary, being defined by the almost constant position of the moon. Thus, if one could imagine the earth completely girdled by water, the depth of the water as measured from a point fixed to the earth's solid surface would pass through two maxima and two minima in each revolution. A better approximation to the observed facts is obtained by considering the bulges to be dragged eastward by friction from the land and the ocean floor, so that their equilibrium positions with respect to the moon are more nearly as indicated in Fig. 12–26(b).

To conclude these preliminary remarks, we may point out that the bulges are, in fact, also being carried slowly eastward all the time by the moon's own motion around the earth. This motion (one complete orbit relative to the fixed stars every

Fig. 12–26 (a) Double tidal bulge as it would be if the earth's rotation did not displace it. The size of the bulge is enormously exaggerated. (b) Approximate true orientation of the tidal bulges, carried eastward by the earth's rotation.



27.3 days) has the consequence that it takes more than 24 hr for a given point on the earth to make successive passages past a particular tidal bulge. Specifically, this causes the theoretical time interval between successive high tides at a given place to be close to 12 hr 25 min instead of precisely 12 hr (see Problem 2–15). For example, if a high tide is observed to occur at 4 P.M. one day, its counterpart next day would be expected to occur at about 4:50 P.M.

Now let us consider the dynamical situation. The first point to appreciate is the manner in which the earth as a whole is being accelerated toward the moon by virtue of the gravitational attraction between them. With respect to the CM of the earth–moon system (inside the earth, at about 3000 miles from the earth's center), the earth's center of mass has an acceleration of magnitude a_C given by Newton's laws:

$$M_E a_C = \frac{G M_E M_m}{r_m{}^2}$$

i.e.,

$$a_C = \frac{G M_m}{r_m{}^2} \qquad (12\text{–}19)$$

where M_m and r_m are the moon's mass and distance. What may not be immediately apparent is that every point in the earth receives this *same* acceleration from the moon's attraction. If

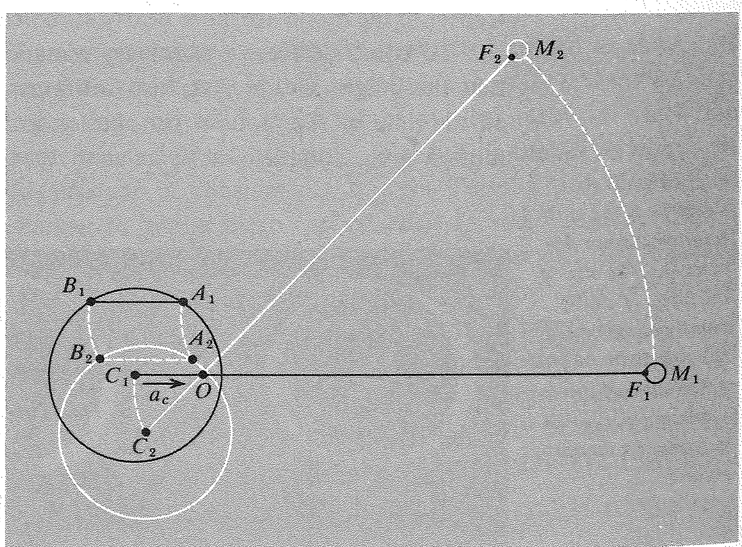


Fig. 12–27 The orbital motion of the earth about the moon does not by itself involve any rotation of the earth; the line A_1B_1 is carried into the parallel configuration A_2B_2.

one draws a sketch, as shown in Fig. 12–27, of the arcs along which the earth's center and the moon travel in a certain span of time, one is tempted to think of the earth–moon system as a kind of rigid dumbbell that rotates as a unit about the center of mass, O. It is true that the moon, for its part, does move so that it presents always the same face toward the earth, but with the earth itself things are different. If the earth were not rotating on its axis, every point on it would follow a circular arc identical in size and direction to the arc C_1C_2 traced out by the earth's center. The line A_1B_1 would be translated into the parallel line A_2B_2. The earth's intrinsic rotation about its axis is simply superposed on this general displacement and the associated acceleration.

This is where noninertial frames come into the picture. The dynamical consequences of the earth's orbital motion around the CM of the earth–moon system can be correctly described in terms of an inertial force, $-ma_C$, experienced by a particle of mass m wherever it may be, in or on the earth. This force is then added to all the other forces that may be acting on the particle.

In the model that we are using—corresponding to what is called the *equilibrium theory* of the tides—the water around the earth simply moves until it attains an equilibrium configuration that remains stationary from the viewpoint of an observer on the moon. Now we know that for a particle at the earth's center, the centrifugal force and the moon's gravitational attraction are equal and opposite. If, however, we consider a particle on the earth's surface at the nearest point to the moon [point A in Fig. 12–28(a)], the gravitational force on it is greater than the centrifugal force by an amount that we shall call f_0:

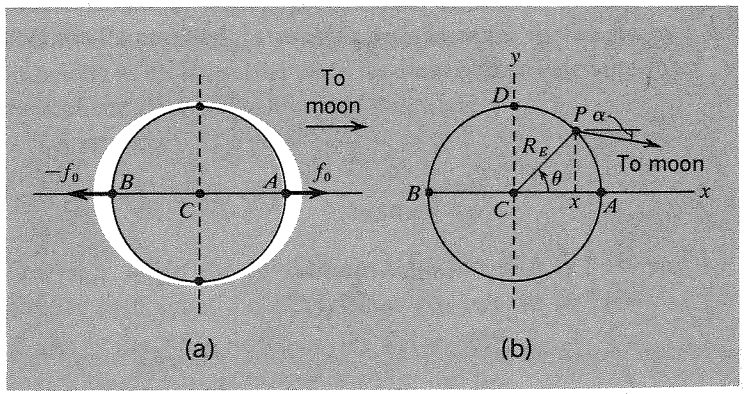


Fig. 12–28 (a) Difference between centrifugal force and the earth's gravity at the points nearest to and farthest from the moon. (b) Tide-producing force at an arbitrary point P, showing existence of a transverse component.

$$f_0 = \frac{GM_m m}{(r_m - R_E)^2} - \frac{GM_m m}{r_m^2}$$

Since $R_E \ll r_m$ ($R_E \approx r_m/60$), we can approximate this expression as follows:

$$f_0 = \frac{GM_m m}{r_m^2}\left[\left(1 - \frac{R_E}{r_m}\right)^{-2} - 1\right]$$

i.e.,

$$f_0 \approx \frac{2GM_m m}{r_m^3} R_E \qquad (12\text{–}20)$$

By an exactly similar calculation, we find that the tide-producing force on a particle of mass m at the farthest point from the moon [point B in Fig. 12–28(a)] is equal to $-f_0$; hence we recognize the tendency for the water to be pulled or pushed away from a midplane drawn through the earth's center (see the figure).

By going just a little further we can get a much better insight into the problem. Consider now a particle of water at an arbitrary point P [Fig. 12–28(b)]. Relative to the earth's center, C, it has coordinates (x, y), with $x = R_E \cos\theta$, $y = R_E \sin\theta$. The tidal force on it in the x direction is given by a calculation just like those above:

$$f_x \approx \frac{2GM_m m}{r_m^3} x = \frac{2GM_m m}{r_m^3} R_E \cos\theta \qquad (12\text{–}21)$$

This yields the results already obtained for the points A and B if we put $\theta = 0$ or π. In addition to this force parallel to the line joining the centers of the earth and the moon there is also, however, a transverse force, because the line from P to the moon's center makes a small angle, α, with the x axis, and the net gravitational force, $GM_m m/r^2$, has a small component perpendicular to x, given by

$$f_y = -\frac{GM_m m}{r^2} \sin\alpha \qquad (\text{with } r \approx r_m)$$

Now we have

$$\tan\alpha = \frac{y}{r_m - x}$$

Since α is a very small angle [$\leq \tan^{-1}(R_E/r_m)$, which is about 1°] we can safely approximate the above expression:

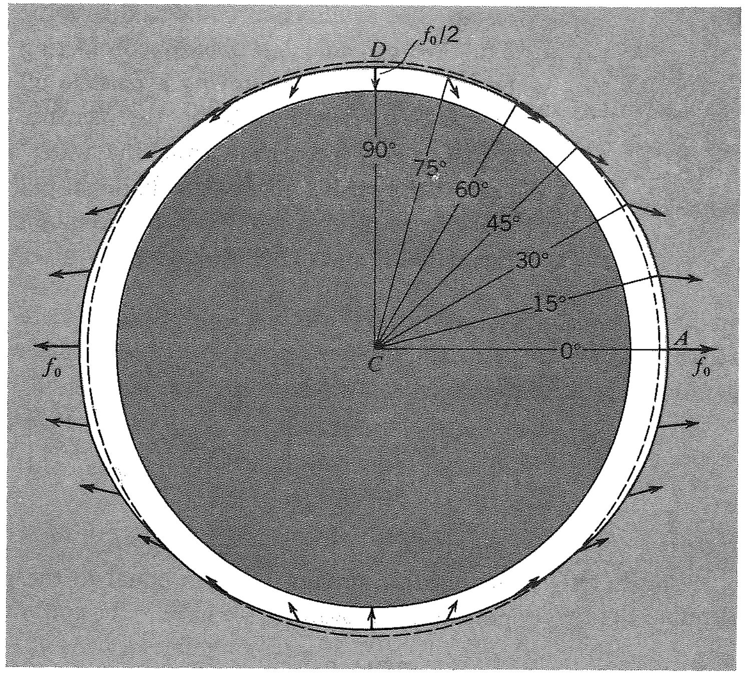


Fig. 12–29 Pattern of tide-producing forces around the earth. The circular dashed line shows where the undisturbed water surface would be.

$$\alpha \approx \frac{y}{r_m} = \frac{R_E \sin\theta}{r_m}$$

The component f_y of the tidal force is then given by

$$f_y \approx \frac{GM_m m}{r_m^3} y = -\frac{GM_m m}{r_m^3} R_E \sin\theta \qquad (12\text{–}22)$$

We see that this transverse force is greatest at $\theta = \pi/2$, at which point it is equal to half the maximum value (f_0) of f_x. Using Eqs. (12–21) and (12–22) together, we can develop an over-all picture of the tide-producing forces, as shown in Fig. 12–29. This shows much more convincingly how the forces act in such directions as to cause the water to flow and redistribute itself in the manner already qualitatively described.

TIDAL HEIGHTS; EFFECT OF THE SUN[1]

How high ought the equilibrium tidal bulge to be? If you are familiar with actual tidal variations you may be surprised at the

[1]This section goes well beyond the scope of the chapter as a whole but is added for the interest that it may have.

result. The equilibrium tide would be a rise and fall of less than 2 ft. We can calculate this by considering that the work done by the tidal force in moving a particle of water from D to A (Fig. 12–29) is equivalent to the increase of gravitational potential energy needed to raise the water through a height h against the earth's normal gravitational pull.[1] The distance h is the difference of water levels between A and D. Now, using Eqs. (12–21) and (12–22) we have

$$dW = f_x\,dx + f_y\,dy$$
$$= \frac{GM_m m}{r_m^3}(2x\,dx - y\,dy)$$
$$W_{DA} = \frac{GM_m m}{r_m^3}\left[\int_0^{R_E} 2x\,dx - \int_{R_E}^{0} y\,dy\right]$$
$$= \frac{3GM_m m}{2r_m^3} R_E^2$$

Setting this amount of work equal to the gain of gravitational potential energy, mgh, we have

$$h = \frac{3GM_m R_E^2}{2gr_m^3} \qquad (12\text{–}23)$$

The numerical values of the relevant quantities are as follows:

$$G = 6.67 \times 10^{-11}\ \text{m}^3/\text{kg-sec}^2$$
$$M_m = 7.34 \times 10^{22}\ \text{kg}$$
$$r_m = 3.84 \times 10^8\ \text{m}$$
$$R_E = 6.37 \times 10^6\ \text{m}$$
$$g = 9.80\ \text{m/sec}^2$$

Substituting these in Eq. (12–23) we find

$$h \approx 0.54\ \text{m} \approx 21\ \text{in.}$$

The great excess over this calculated value in many places (by factors of 10 or even more) can only be explained by considering the problem in detailed dynamical terms, in which the accumulation of water in narrow estuaries, and resonance effects, can completely alter the scale of the phenomenon. The value that we have calculated should be approximated in the open sea.

The last point that we shall consider here is the effect of the sun. Its mass and distance are as follows:

[1]Technically, this condition corresponds to the water surface being an energy equipotential.

$$M_s = 1.99 \times 10^{30}\ \text{kg}$$
$$r_s = 1.49 \times 10^{11}\ \text{m}$$

If we directly compare the gravitational forces exerted by the sun and the moon on a particle on the earth, we discover that the sun wins by a large factor:

$$\frac{F_s}{F_m} = \frac{M_s/r_s^2}{M_m/r_m^2} = \frac{M_s}{M_m}\left(\frac{r_m}{r_s}\right)^2 \approx 180$$

What matters, however, for tide production is the amount by which these forces *change* from point to point over the earth. This is expressed in terms of the *gradient* of the gravitational force:

$$F(r) = \frac{GMm}{r^2}$$
$$f = \Delta F = -\frac{2GMm}{r^3}\Delta r \qquad (12\text{–}24)$$

Putting $M = M_m$, $r = r_m$, and $\Delta r = \pm R_E$, we obtain the forces $\pm f_0$ corresponding to Eq. (12–20).

We now see that the comparative tide-producing forces due to the sun and the moon are given, according to Eq. (12–24), by the following ratio:

$$\frac{f_s}{f_m} = \frac{M_s/r_s^3}{M_m/r_m^3} = \frac{M_s}{M_m}\left(\frac{r_m}{r_s}\right)^3 \qquad (12\text{–}25)$$

Substituting the numerical values, one finds

$$\frac{f_s}{f_m} \approx 0.465$$

This means that the tide-raising ability of the moon exceeds that of the sun by a factor of about 2.15. The effects of the two combine linearly—and, of course, vectorially, depending on the relative angular positions of the moon and the sun. When they are on the same line through the earth (whether on the same side or on opposite sides) there should be a maximum tide equal to 1.465 times that due to the moon alone. This should happen once every 2 weeks, approximately, when the moon is new or full. At intermediate times (half-moon) when the angular positions of sun and moon are separated by 90°, the tidal amplitude should fall to a minimum value equal to 0.535 times that of the moon. The ratio of maximum to minimum values is thus about 2.7.

The phenomena that we have discussed in this chapter seem to leave us in no doubt that the acceleration of one's frame of reference can be detected by dynamical means. They suggest that a very special status does indeed attach to inertial frames. But how can we be sure that we have identified a true inertial frame in which Galileo's law of inertia holds exactly?

We saw at the very beginning of our discussion of dynamics that the earth itself represents a good approximation to such a frame for many purposes, especially for dynamical phenomena whose scale in distance and time is small. But we have now seen abundant evidence that a laboratory on the earth's surface is accelerated. If the laboratory is at latitude λ (see Fig. 12–30), each point in it is accelerating toward the earth's axis of rotation with an acceleration given by

$$a_\lambda = \omega^2 R \cos\lambda$$

with

$$\omega = 2\pi/86{,}400\ \text{sec}^{-1}$$
$$R = 6.4 \times 10^6\ \text{m}$$

This gives

$$a_\lambda = 3.4 \times 10^{-2} \cos\lambda\ \text{m/sec}^2$$

This acceleration of a frame of reference tied to the earth is, as we know, not the simplest case of an accelerated frame. The linearly accelerated frames with which we began this chapter are much more readily analyzed. It was, however, the phenomena associated with rotating frames that led Newton to his belief in absolute space and in the absolute character of accelerations. Near the beginning of the *Principia* he describes a celebrated

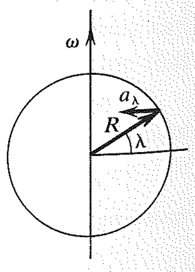


Fig. 12–30 Acceleration toward the earth's axis by virtue of its rotation.

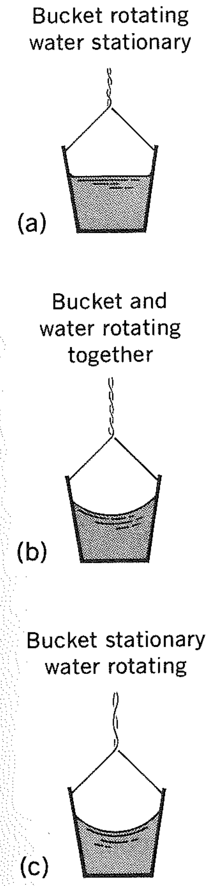


Fig. 12–31 Main features of the experiment that Newton quoted as evidence of the absolute character of rotation and the associated acceleration.

experiment that he made with a bucket of water. It is an experiment that anyone may readily repeat for himself. The bucket is hung on a strongly twisted rope and is then released. There are three key observations, depicted in Fig. 12–31:

1. At first the bucket spins rapidly, but the water remains almost at rest, before the viscous forces have had time to set it rotating. The water surface is flat, just as it was before the bucket was released.

2. The water and the bucket are rotating together; the water surface has become concave (see Problem 12–18).

3. The bucket is suddenly stopped, but the body of water continues to rotate, and its surface remains curved.[1]

Clearly, said Newton, the relative motion of the bucket and the water is not the factor that determines the curvature of the water surface. It must be the absolute rotation of the water in space, and its attendant acceleration, that is at the bottom of the phenomenon. And with the help of $F = ma$, we can account for it quantitatively.

Newton's argument is a powerful one. He could point to further evidence in support of his views in the bulging of the earth itself by virtue of its rotation. The equatorial diameter of the earth is greater than the polar diameter by about 1 part in 300. It seems almost obvious, even without detailed calculation, that this is closely tied to the fact that a_λ/g is about $\frac{1}{300}$ at the equator and is zero at the poles (although the detailed calculation is, in fact, a bit messy).

Newton did not stop here, of course, He held the key of universal gravitation. Even a nonrotating earth would not be an inertial frame, because the whole earth is accelerating toward the sun.

For this system we have

$$\omega = 2\pi/(3.16 \times 10^7)\ \text{sec}^{-1}$$
$$R = 1.49 \times 10^{11}\ \text{m}$$
$$a_2 = \omega^2 R = 5.9 \times 10^{-3}\ \text{m/sec}^2$$

[1]Newton does not suggest that he actually performed this third step, but it represents a natural completion of the experiment as one might perform it for oneself.

If we could conceive of an object that was immune to the gravitational attraction of the sun, it would not obey the law of inertia as observed from a reference frame attached to the earth. From Newton's standpoint the acceleration is real and absolute and is linked to the existence of a well-defined gravitational force provided by the sun.

That was about the end of the road as far as Newton was concerned. For him the system of the stars provided the arena in which the motions that he so brilliantly analyzed took place. A reference frame attached to these fixed stars could be taken to constitute a true inertial system, even though it might not coincide with the absolute space in which he believed.

Today, thanks to the work of astronomers, we know a good deal about the motions of some of those "fixed" stars. We have come to be aware of our involvement in a general rotation of our Galaxy. The sun would appear to be making a complete circuit of the Galaxy in about 2.5×10^8 years at a radial distance of about 2.5×10^4 light-years from the center. For this motion we would have

$$\omega \approx 2\pi/(8 \times 10^{15})\ \text{sec}^{-1}$$
$$R \approx 2.4 \times 10^{20}\ \text{m}$$
$$a_3 \approx 10^{-10}\ \text{m/sec}^2$$

It looks as though this acceleration can be reasonably accounted for by means of Newton's law of universal gravitation, if we regard the solar system as having a centripetal acceleration under the attraction of all the stars lying within its orbit. But no dynamical experiments that we do on earth require us to take into account this extremely minute effect—or, even, for most purposes, the revolution of the earth about the sun. (The rotation of the earth on its own axis is, however, an important consideration—and indeed an important aid in such matters as gyroscopic navigation.) Figure 12–32 schematizes the three rotating frames in which we find ourselves (we ignore here the acceleration caused by the moon).

But we still have not found an unaccelerated object to which we can attach our inertial frame of reference. In fact, we could extend this tantalizing search even further. There is some evidence that galaxies themselves tend to cluster together in groups containing a few galaxies to perhaps thousands. Our local group consists of about 10 galaxies. Although individual galaxies could have rather complex motions with respect to each other,

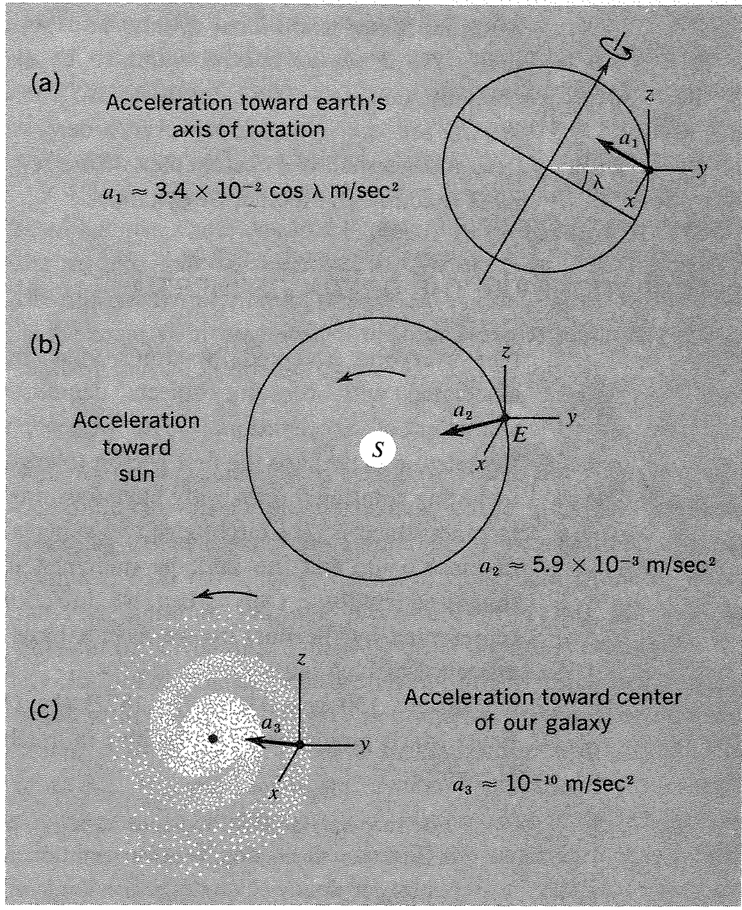


Fig. 12–32 Accelerations of any laboratory reference frame attached to the earth's surface.

this group is believed to have a more or less common motion through space.

So where are the "fixed" stars or other astronomical objects to which we can attach our inertial frame of reference? It appears that referring to the "fixed stars" is not a solution and contains an uncomfortable element of metaphysics (although we frequently use this phrase as a shorthand designation for the establishment of an inertial frame). This does not mean that the astronomical search for an inertial frame has been without value. For, at least up to the galactic level, it would seem that apparent departures from the law of inertia can be traced to identifiable accelerations of the reference frame in which motions are observed. However, the quest is incomplete, and so it seems likely to remain. Ultimately, therefore, we rely on an operational definition based upon local dynamical experiments and observation. We *define* an inertial frame to be one in which, experimentally, Galileo's law of inertia holds. The very existence of the inertial property remains, however, a deep and fascinating problem, and we shall end the chapter with a few remarks about this most fundamental feature of dynamics.

SPECULATIONS ON THE ORIGIN OF INERTIA

Not everyone accepted Newton's view that the phenomena associated with rotating objects demonstrated the absolute character of acceleration. The philosopher–bishop, George Berkeley, was perhaps the first person to argue[1] that all motions, including rotational ones, only have meaning as motions relative to other objects. The circling of two spheres around their center of mass could not, he said, be imagined in a space that was otherwise empty. Only when we introduce the background represented by the stars do we have a basis for recognizing the existence of such motion.

About 150 years later (in 1872) the German philosopher Ernst Mach presented the same idea in much more cogent form. He wrote:

> For me, only relative motions exist . . . and I can see, in this regard, no distinction between rotation and translation. Obviously it does not matter if we think of the earth as turning round on its axis, or at rest while the fixed stars revolve around it . . . But if we think of the earth at rest and the fixed stars revolving around it, there is no flattening of the earth, no Foucault's experiment and so on—at least according to our usual conception of the law of inertia. Now one can solve the difficulty in two ways. Either all motion is absolute, or our law of inertia is wrongly expressed . . . I prefer the second way. The law of inertia must be so conceived that exactly the same thing results from the second supposition as from the first. By this it will be evident that in its expression, regard must be paid to the masses of the universe.[2]

[1]In his tract *De Motu*, written in 1717, 30 years after the publication of Newton's *Principia*.

[2]E. Mach, *History and Root of the Principle of the Conservation of Energy*, (2nd ed.), Barth, Leipzig (1909). English translation of the 2nd edition by P. Jourdain, Open Court Publishing Co., London, 1911. Actually the first sentence of the quotation is taken from Mach's classic book, *The Science of Mechanics*, first published in 1883.

Thus was born the profound and novel idea—subsequently to become famous as *Mach's principle*—that the inertial property of any given object depends upon the presence and the distribution of other masses. Einstein himself accepted this idea and took it as a central principle of cosmology.

If one admits the validity of this point of view, then one sees that the whole basis of dynamics is involved. For consider the method that we described in Chapter 9 (p. 319) for finding the ratio of the inertial masses of two objects. This ratio is given as the negative inverse ratio of the accelerations that they produce by their mutual interaction:

$$\frac{m_1}{m_2} = -\frac{a_2}{a_1}$$

This looks very simple and straightforward, but it is clear that our ability to attach specific values to the individual accelerations, as distinct from the total *relative* acceleration, depends completely on our having identified a reference frame in which these accelerations can be measured. For this purpose the physical background provided by other objects is essential.

In looking critically at the phenomena of rotational motion, Mach attacked some intuitive notions that are much more deep-seated than any that we have in connection with straight-line motion. He considered the evidence provided by Newton's rotating-bucket experiment which we discussed in the last section. It is quite clear that the curvature of the water surface is related overwhelmingly to the existence of rotation relative to the vast amount of distant matter of the universe. When that relative rotation is stopped, the water surface becomes flat. When the bucket rotates and the water remains still (both relative to the fixed stars), the shape of the water surface remains unaffected. But, said Mach, that may be only a matter of degree. "No one," he wrote, "is competent to say how the experiment would turn out if the sides of the vessel increased in thickness and mass until they were ultimately several leagues thick." His own belief was that this rotation of a monster bucket would in fact generate the equivalent of centrifugal forces on the water inside it, even though this water had no rotational motion in the accepted sense.

This is a startling idea indeed. Let us present it in a slightly different context. We know that the act of giving an object an acceleration **a**, with respect to the inertial frame defined by the fixed stars, calls into play an inertial force, equal to $-m\mathbf{a}$, that expresses the resistance of the object to being accelerated. In

Mach's view we are equally entitled (indeed, compelled) to accept a description of the phenomenon in a frame always attached to the object itself. In this frame the rest of the universe has the acceleration $\mathbf{a}'$ ($= -\mathbf{a}$) and the inertial force $m\mathbf{a}'$ that the object experiences must be ascribable to the acceleration of the other masses.

This then brings us to the quantitative question: If a mass M, at distance r, is given the acceleration $\mathbf{a}$ relative to a given object, what contribution does it make to the total inertial force $m\mathbf{a}$ that the object experiences? Since we know that the force is proportional to m, we can argue on the grounds of symmetry and relativity that it must be proportional to M also. But at this point we enter a more speculative realm. A very suggestive analogy is provided by electromagnetic interactions. If two electric charges, q_1 and q_2, are separated by a distance r, we know that the static force exerted by q_1 on q_2 is given by

$$F_{12} = \frac{kq_1q_2}{r^2}$$

where k is a constant that depends on the particular choice of units. If, however, the charge q_1 is given the acceleration a there is an additional force that comes into play, directly proportional to a and inversely proportional to the distance:

$$F'_{12} = \frac{kq_1q_2a}{c^2r}$$

where c is the speed of light. Since this force falls off more slowly with distance than the static interaction, it can survive in appreciable magnitude at distances at which the static $1/r^2$ force has become negligible. This is, in fact, the basis of the electromagnetic radiation field by which signals can be transmitted over large distances.

Suppose now that we assume an analogous situation for gravitational interactions. The basic static law of force is known to be

$$F_{12} = \frac{GMm}{r^2}$$

The force on m associated with an acceleration of M would then be given by

$$F'_{12} = \frac{GMma}{c^2r} \qquad (12\text{–}26)$$

On this basis we can estimate the relative magnitudes of the contributions from various masses of interest—the earth, the sun, our own Galaxy, and the rest of the universe. All we have to do is to calculate the values of M/r for these objects. The results are shown in Table 12–1, using numbers to the nearest power of 10 only. (The value of M for the universe as a whole is the somewhat speculative value quoted in Chapter 1.) We

TABLE 12–1: RELATIVE CONTRIBUTIONS TO INERTIA

Source	M, kg	r, m	M/r, kg/m	M/r (relative)
Earth	10^{25}	10^{7}	10^{18}	10^{-8}
Sun	10^{30}	10^{11}	10^{19}	10^{-7}
Our Galaxy	10^{41}	10^{21}	10^{20}	10^{-6}
Universe	10^{52}	10^{26}	10^{26}	1

see that, according to this theory, the effect of a nearby object, even one as massive as the earth itself, would be negligible compared to the effect of the universe at large.

The *total* inertial force called into existence if everything in the universe acquires an acceleration a with respect to a given object would be obtained by summing the forces F'_{12} of Eq. (12–26) over all masses other than m itself:

$$F_{\text{inertial}} = ma\sum\frac{GM}{c^2r}$$

This, however, should be identical with what we know to be the magnitude of the inertial force as directly given by the value of ma. Thus the theory would require the following identity to hold:

$$\sum_{\text{universe}}\frac{GM}{c^2r} = 1 \qquad (12\text{–}27)$$

It is clear from Table 12–1 that even such a large local mass as our Galaxy represents only a minor contribution; what we are involved with is a summation over the approximately uniform distribution of matter represented by the universe as a whole. If we regard it as a sphere, centered on ourselves, of mean density ρ and radius R_U ($\approx 10^{10}$ light-years $= 10^{26}$ m), we would have

$$\sum_{\text{universe}}\frac{M}{r} \rightarrow \int_0^{R_U}\frac{4\pi\rho r^2\,dr}{r} = 2\pi\rho R_U{}^2$$

The total mass is however given by

$$M_U = \frac{4\pi}{3}\rho R_U^3$$

Thus we have, on this simple picture (based on Euclidean geometry)

$$\sum_{\text{universe}} \frac{M}{r} = \frac{3}{2}\frac{M_U}{R_U} \approx 10^{26}\ \text{kg/m}$$

Using the values $G \approx 10^{-10}$ N-m^2/kg^2 and $c^2 \approx 10^{17}$ m^2/sec^2, we would then have

$$\sum_{\text{universe}} \frac{GM}{c^2 r} \approx 10^{-1}$$

Taking into account the uncertainties in our knowledge of the distribution of matter throughout space, many would say that the factor of about 10 that separates the above empirical value from the theoretical value (unity) called for by Eq. (12–27) is not significant. The result is intriguing, to say the least, and many cosmologists have accepted as fundamentally correct this development from the primary ideas espoused by Mach and Einstein.[1]

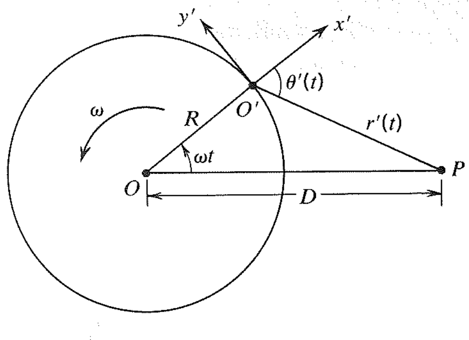
PROBLEMS

12–1 A single-engine airplane flies horizontally at a constant speed v. In the frame of the aircraft, each tip of the propeller sweeps out a circle of radius R at the rate of n revolutions per second. Obtain an equation for the path of a tip of the propeller as viewed from the earth.

12–2 A person observes the position of a post from the origin of a reference frame (S') rigidly attached to the rim of a merry-go-round, as shown in the figure. The merry-go-round (of radius R) is rotating with angular velocity ω, the distance of the post from the axis of the merry-go-round is D, and at $t = 0$, the coordinates of P in S' are $x' = D - R$, $y' = 0$ (equivalently, $r' = D - R$, $\theta' = 0$).

(a) Find the coordinates $r'(t)$, $\theta'(t)$ of the post; also give the corresponding $x'(t)$ and $y'(t)$.

[1]For further reading on this fascinating topic, see, for example, R. H. Dicke, "The Many Faces of Mach," in *Gravitation and Relativity* (ed. H.-Y. Chin and W. F. Hoffmann, eds.), W. A. Benjamin, New York, 1964; N. R. Hanson, "Newton's First Law," and P. Morrison, "The Physics of the Large," both in *Beyond the Edge of Certainty* (R. G. Colodny, ed.), Prentice-Hall, Englewood Cliffs, N.J., 1965; D. W. Sciama, *The Unity of the Universe*, Doubleday, New York, 1961, and *The Physical Foundations of General Relativity*, Doubleday, New York, 1969.



(b) By differentiating the results of (a), obtain the velocity and acceleration of the post in both Cartesian and polar coordinates.

(c) Make a plot of the path of the post in S'.

12–3 A boy is riding on a railroad flatcar, on level ground, that has an acceleration a in the direction of its motion. At what angle with the vertical should he toss a ball so that he can catch it without shifting his position on the car?

12–4 A railroad train traveling on a straight track at a speed of 20 m/sec begins to slow down uniformly as it enters a station and comes to a stop in 100 m. A suitcase of mass 10 kg having a coefficient of sliding friction $\mu = 0.15$ with the train's floor slides down the aisle during this deceleration period.

(a) What is the acceleration of the suitcase (with respect to the ground) during this time?

(b) What is the velocity of the suitcase just as the train comes to a halt?

(c) The suitcase continues sliding for a period after the train has stopped. When it comes to rest, how far is it displaced from its original position on the floor of the train?

12–5 A man weighs himself on a spring balance calibrated in newtons which indicates his weight as $mg = 700$ N. What will he read if he repeats the observation while riding an elevator from the first to the twelfth floors in the following manner?

(a) Between the first and third floors the elevator accelerates at the rate of 2 m/sec^2.

(b) Between the third and tenth floors the elevator travels with the constant velocity of 7 m/sec.

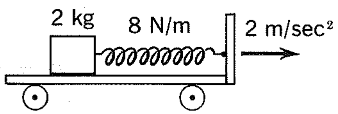
(c) Between the tenth and twelfth floors the elevator decelerates at the rate of 2 m/sec^2.

(d) He then makes a similar trip down again.

(e) If on another trip the balance reads 500 N, what can you say of his motion? Which way is he moving?

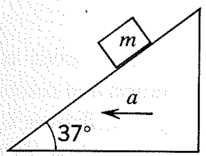
12–6 If the coefficient of friction between a box and the bed of a truck is μ, what is the maximum acceleration with which the truck can climb a hill, making an angle θ with the horizontal, without the box's slipping on the truck bed?

12–7 A block of mass 2 kg rests on a frictionless platform. It is attached to a horizontal spring of spring constant 8 N/m, as shown in the figure. Initially the whole system is stationary, but at $t = 0$ the platform begins to move to the right with a constant acceleration of 2 m/sec.² As a result the block begins to oscillate horizontally relative to the platform.



(a) What is the amplitude of the oscillation?

(b) At $t = 2\pi/3$ sec, by what amount is the spring longer than it was in its initial unstretched condition?



12–8 A plane surface inclined 37° ($\sin^{-1}\frac{3}{5}$) from the horizontal is accelerated horizontally to the left (see the figure). The magnitude of the acceleration is gradually increased until a block of mass m, originally at rest with respect to the plane, just starts to slip up the plane. The static friction force at the block-plane surface is characterized by $\mu = \frac{4}{5}$.

(a) Draw a diagram showing the forces acting on the block, just before it slips, in an inertial frame fixed to the floor.

(b) Find the acceleration at which the block begins to slip.

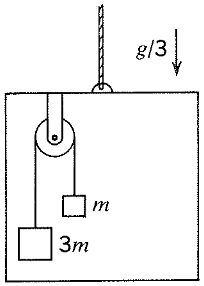
(c) Repeat part (a) in the noninertial frame moving along with the block.

12–9 A nervous passenger in an airplane at takeoff removes his tie and lets it hang loosely from his fingers. He observes that during the takeoff run, which lasts 30 sec, the tie makes an angle of 15° with the vertical. What is the speed of the plane at takeoff, and how much runway is needed? Assume that the runway is level.

12–10 A uniform steel rod (density = 7500 kg/m³, ultimate tensile strength 5×10^8 N/m²) of length 1 m is accelerated along the direction of its length by a constant force applied to one end and directed away from the center of mass of the rod. What is the maximum allowable acceleration if the rod is not to break? If this acceleration is exceeded, where will the rod break?

12–11 (a) A train slowed with deceleration a. What angle would the liquid level of a bowl of soup in the dining car have made with the horizontal? A child dropped an apple from a height h and a distance d from the front wall of the dining car. What path did the apple take as observed by the child? Under what conditions would the apple have hit the ground? The front wall?

(b) As a reward for making the above observations, the parents bought the child a helium-filled balloon at the next stop. For fun, they asked him what would happen to the balloon if the train left the station with acceleration a'. Subsequently, they were surprised to find his predictions correct. What did the precocious child answer?



12–12 An elevator has a *downward* acceleration equal to $g/3$. Inside the elevator is mounted a pulley, of negligible friction and inertia, over which passes a string carrying two objects, of masses m and $3m$, respectively (see the figure).

(a) Calculate the acceleration of the object of mass $3m$ *relative to the elevator*.

(b) Calculate the force exerted on the pulley by the rod that joins it to the roof of the elevator.

(c) How could an observer, completely isolated inside the elevator, explain the acceleration of m in terms of forces that he himself could measure with the help of a spring balance?

12–13 In each of the following cases, find the equilibrium position as well as the period of small oscillations of a pendulum of length L:

(1) In a train moving with acceleration a on level tracks.

(2) In a train free-wheeling on tracks making an angle θ with the horizontal.

(3) In an elevator falling with acceleration a.

12–14 The world record for the 16-lb hammer throw is about 70 m. Assuming that the hammer is whirled around in a circle of radius about 2 m before being let fly, estimate the magnitude of the pull that the thrower must be able to withstand.

12–15 (a) A man rides in an elevator with vertical acceleration a. He swings a bucket of water in a vertical circle of radius R. With what angular velocity must he swing the bucket so that no water spills?

(b) With what angular frequency must the bucket be swung if the man is on a train with horizontal acceleration a? (The plane of the circle is again vertical and contains the direction of the train's acceleration.)

12–16 Consider a thin rod of material of density ρ rotating with constant angular velocity ω about an axis perpendicular to its length.

(a) Show that if the rod is to have a constant stress S (tensile force per unit area of cross section) along its length, the cross-sectional area must decrease exponentially with the square of the distance from the axis:

$$A = A_0 e^{-kr^2} \qquad \text{where } k = \rho\omega^2/2S$$

[Consider a small segment of the rod between r and $r + \Delta r$, having a mass $\Delta m = \rho A(r)\,\Delta r$, and notice that the difference in tensions at its ends is $\Delta T = \Delta(SA)$.]

(b) What is the maximum angular velocity ω_{max} in terms of ρ, S_{max}, and k?

(c) The ultimate tensile strength of steel is about 10^9 N/m². Estimate the maximum possible number of rpm of a steel rotor for which the "taper constant" $k = 100$ m^{-2} ($\rho = 7500$ kg/m³).

12–17 A spherically shaped influenza virus particle, of mass 6×10^{-16} g and diameter 10^{-5} cm, is in a water suspension in an ultracentrifuge. It is 4 cm from the vertical axis of rotation, and the speed of rotation is 10^3 rps. The density of the virus particle is 1.1 times that of water.

(a) From the standpoint of a reference frame rotating with the centrifuge, what is the effective value of "g"?

(b) Again from the standpoint of the rotating reference frame, what is the net centrifugal force acting on the virus particle?

(c) Because of this centrifugal force, the particle moves radially outward at a small speed v. The motion is resisted by a viscous force given by $F_{res} = 3\pi\eta v d$, where d is the diameter of the particle and η is the viscosity of water, equal to 10^{-2} cgs units (g/cm/sec). What is v?

(d) Describe the situation from the standpoint of an inertial frame attached to the laboratory.

[In (b) and (c), account must be taken of buoyancy effects. Think of the ordinary hydrostatics problem of a body completely immersed in a fluid of different density.]

12–18 (a) Show that the equilibrium form of the surface of a rotating body of liquid is parabolic (or, strictly, a paraboloid of revolution). This problem is most simply considered from the standpoint of the rotating frame, given that a liquid cannot withstand forces tangential to its surface and will tend toward a configuration in which such forces disappear. It is instructive to consider the situation from the standpoint of an inertial frame also.

(b) It has been proposed that a parabolic mirror for an astronomical telescope might be formed from a rotating pool of mercury. What rate of rotation (rpm) would make a mirror of focal length 20 m?

12–19 To a first approximation, an object released anywhere within an orbiting spacecraft will remain in the same place relative to the spacecraft. More accurately, however, it experiences a net force proportional to its distance from the center of mass of the spacecraft. This force, as measured in the noninertial frame of the craft, arises from the small variations in both the gravitational force and the centrifugal force due to the change of distance from the earth's center. Obtain an expression for this force as a function of the mass, m, of the object, its distance ΔR from the center of the spacecraft, the radius R of the spacecraft's orbit around the earth, and the gravitational acceleration g_R at the distance R from the earth's center.

12–20 A circular platform of radius 5 m rotates with an angular velocity $\omega = 0.2$ rad/sec. A man of mass 100 kg walks with constant velocity $v' = 1$ m/sec along a diameter of the platform. At time $t = 0$ he crosses the center and at time $t = 5$ sec he jumps off the edge of the platform.

(a) Draw a graph of the centrifugal force felt by the man as a function of time in the interval $t = 0$ to $t = 5$ sec.

(b) Draw a similar graph of the Coriolis force. For both diagrams, give the correct vertical scale (in newtons).

(c) Show on a sketch the direction of these forces, assuming the platform to rotate in a clockwise direction as seen from above.

12–21 On a long-playing record (33 rpm, 12 in.) an insect starts to crawl toward the rim. Assume that the coefficient of friction between its legs and the record is 0.1. Does it reach the edge by crawling or otherwise?

12–22 A child sits on the ground near a rotating merry-go-round. With respect to a reference frame attached to the earth the child has no acceleration (accept this as being approximately true) and experiences no force. With respect to polar coordinates fixed to the merry-go-round, with origin at its center:

(a) What is the motion of the child?

(b) What is his acceleration?

(c) Account for this acceleration, as measured in the rotating frame, in terms of the centrifugal and Coriolis forces judged to be acting on the child.

12–23 The text (p. 516) derives the Coriolis force in the transverse (θ) direction by considering the motion of an object along a radial line in the rotating frame. Correspondingly, if one considers an object that is moving *transversely* in the rotating frame, one can obtain the net *radial* force due to Coriolis and centrifugal effects. Consider a particle on a frictionless turntable rotating with angular velocity ω. The particle is initially at rest relative to the turntable, at a distance r from the axis of rotation.

(a) Set up a fixed coordinate system S with axes x transversely and y radially and with its origin O at the position of the particle at $t = 0$ (see the figure). Set up another coordinate system S', with origin O' and axes x' and y', which rotates with the turntable and which coincides with S at $t = 0$. Show that at a later time t the coordinates of a given point as measured in S' and S are related by the following equations, where $\theta = \omega t$:

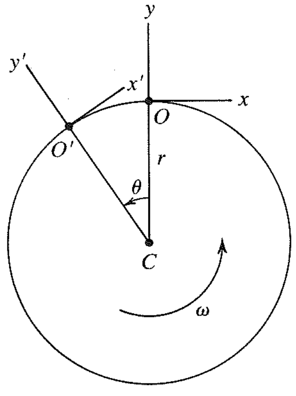
$$x' = x \cos\theta + y \sin\theta + r \sin\theta$$
$$y' = y \cos\theta - x \sin\theta - r(1 - \cos\theta)$$

(b) Suppose that, at $t = 0$, the particle is given a velocity v'

relative to O' in the x' direction. Its subsequent motion will be along the x direction at the constant velocity $v' - \omega r$ relative to O. Use this to obtain its coordinates x' and y' at a later time t.

(c) Making the approximations for the case $\omega t \ll 1$, show that for small values of t one can put $y' \simeq \frac{1}{2}a_r' t^2$, where $a_r' = \omega^2 r - 2\omega v'$. This corresponds to the required combination of centrifugal and Coriolis accelerations.

(d) If you are feeling ambitious, apply the same kind of analysis for an initial velocity in an arbitrary direction.



12–24 In an article entitled "Do Objects fall South?" [*Phys. Rev.*, **16**, 246 (1903)], Edwin Hall reported the results of nearly 1000 trials in which he allowed an object to fall through a vertical distance of 23 m at Cambridge, Mass. (lat. 42° N). He found, on the average, an eastward deflection of 0.149 cm and a southerly deflection of 0.0045 cm.

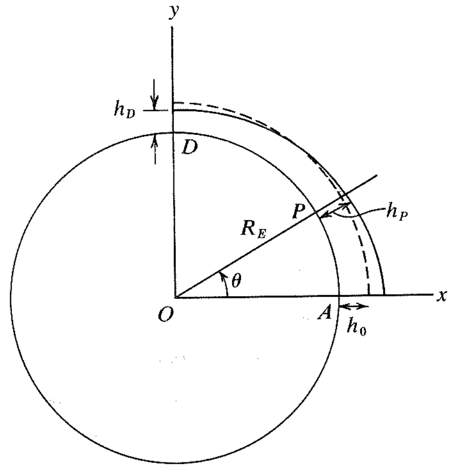
(a) Compare the easterly deflection with what would be expected from Eq. (12–17).

(b) Consider the fact that the development of an eastward component of motion relative to the earth would indeed lead in turn to a southerly component of Coriolis force. Without attempting any detailed analysis, estimate the order of magnitude of the ratio of the resulting southerly deflection to the predominant easterly deflection. Do you think that an explanation of Hall's results on southerly deflection can be achieved in these terms?

12–25 Calculate the Coriolis acceleration of a satellite in a circular polar orbit as observed by someone on the rotating earth. Obtain the direction of this acceleration throughout the orbit, thereby explaining why the satellite always passes through the poles even though it is subjected to the Coriolis force. Is there a similar force on a satellite in an equatorial orbit?

12–26 Imagine that a frictionless horizontal table, circular in shape and of radius R, is fitted with a perfectly elastic rim, and that a dry-ice puck is launched from a point on the rim toward the center. The puck bounces back and forth across the table at constant speed v, but because of the Coriolis force it does not quite follow a straight-line path along a diameter. Consider the rate at which the path of the puck gradually turns with respect to the table, and compare the result with that for a Foucault pendulum at the same latitude, λ.

12–27 In the text (p. 536) the height of the equilibrium tide is calculated by considering the work done by the tide-producing force in carrying a particle of water from point D to point A (see the figure).



By considering the work from D to an intermediate point P, one can obtain a general expression for the elevation or depression $h(\theta)$ of the water at an arbitrary point, relative to what the water level would be in the absence of the tide-producing force. The calculation involves two parts, as follows:

(a) Evaluate the work integral of the tide-producing force from $D(x = 0, y = R_E)$ to $P(x = R_E \cos\theta, y = R_E \sin\theta)$ for a particle of water of mass m. Equating this to the difference of gravitational potential energies, $mg(h_P - h_D)$, one gets an expression for the difference $h_P - h_D$.

(b) The total volume of water is a constant. Hence, if h_0 represents the water depth in the absence of the tide-producing force, we must have

$$\int_0^{\pi/2} 2\pi R_E^2 [h(\theta) - h_0] \sin\theta \, d\theta = 0$$

Putting the results of (a) and (b) together, you should be able to verify that the deviation of the water level from its undisturbed state is proportional to $3\cos^2\theta - 1$.