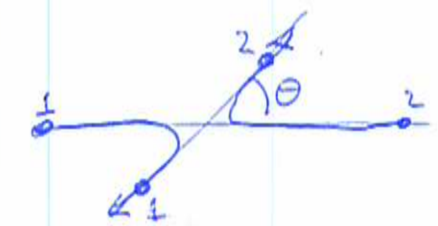
# SYSTEM OF IDENTICAL PARTICLES



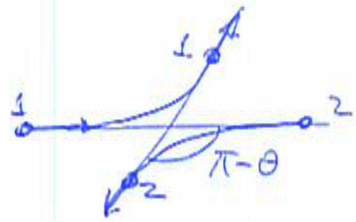
IN CLASSICAL MECHANICS, IDENTICAL PARTICLES ARE DISTINGUISHABLE. ONE CAN LABEL THEM AND FOLLOW THEIR TRAJECTORY

IN QUANTUM MECHANICS, THIS IS NO LONGER TRUE BECAUSE OF THE UNCERTAINTY PRINCIPLE

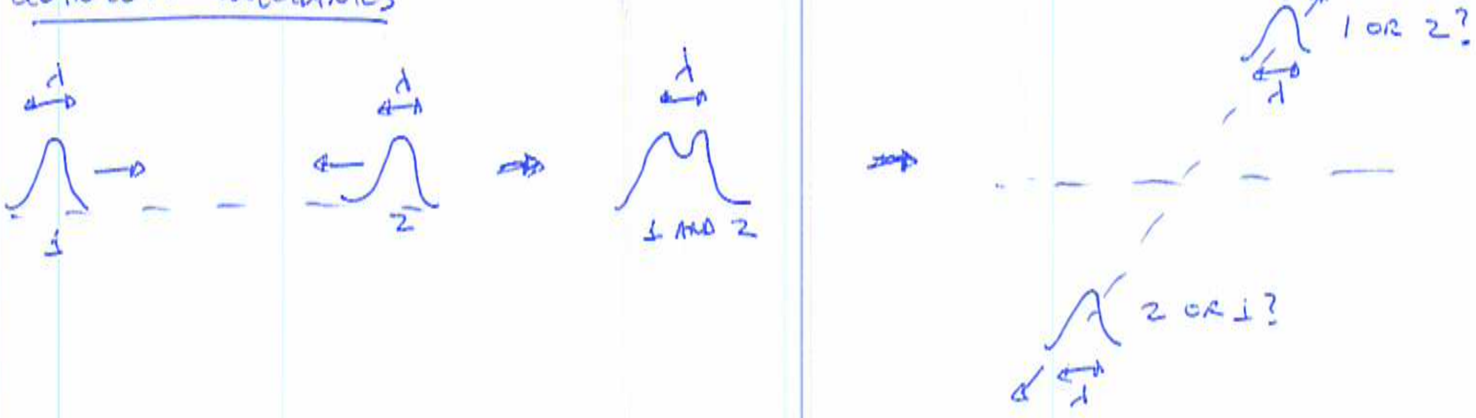
CLASSICAL MECHANICS




IS DIFFERENT FROM




QUANTUM MECHANICS




THUS, IN Q.M., IDENTICAL PARTICLES LOSE THEIR "INDIVIDUALITY" ENTIRELY AND ARE THUS, INDISTINGUISHABLE

CONSIDER A SYSTEM OF N IDENTICAL QUANTUM PARTICLES. THE WAVEFUNCTION IS

$$\psi(q_1, \cdots q_i \cdots q_j \cdots q_N) \quad \text{where}$$

$q_i = (\vec{r}_i, s_i)$, where $s_i$ REPRESENTS THE SPIN

PROJECTION OF THE i-th PARTICLE (2)
AND ALSO ANY OTHER DEGREE OF FREEDOM

LET $P_{ij}$ BE THE PERMUTATION OPERATOR
WHICH EXCHANGE PARTICLES i AND j WITH EACH OTHER

$$\Rightarrow \quad P_{ij}\psi(\cdots q_i \cdots q_j \cdots) = e^{i\gamma}\psi(\cdots q_j \cdots q_i \cdots)$$

SYMMETRIZATION POSTULATE:
BY REPEATING THE EXCHANGE, WE RETURN TO THE ORIGINAL STATE. THUS

$$e^{2i\gamma} = 1 \quad \Rightarrow \quad e^{i\gamma} = \pm 1$$

$$\gamma = 0 \text{ OR } \pi$$

IN SHORT, $\boxed{\psi(\cdots q_i \cdots q_j \cdots) = \pm\psi(\cdots q_j \cdots q_i \cdots)}$

THEREFORE, THE WAVE FUNCTIONS ARE EITHER SYMMETRIC ($\gamma=0$) OR ANTI-SYMMETRIC ($\gamma=\pi$) WHEN A PAIR OF PARTICLES ARE INTERCHANGED

THE PROPERTY OF BEING DESCRIBED BY SYMMETRICAL OR ANTISYMMETRICAL WAVEFUNCTIONS DEPEND ON THE NATURE OF THE PARTICLES:

$\gamma=0$ ARE BOSONS $\to$ OBEY BOSE-EINSTEIN STATISTICS
$\gamma=\pi$ ARE FERMIONS $\to$ " FERMI-DIRAC "

FOR NON-INTERACTING (RELATIVISTIC) QUANTUM FIELD THEORY, PAULI SHOWED THE SPIN-STATISTICS THEOREM [PHYS. REV. 58, 716 (1940)]

- PARTICLES OF INTEGER SPIN ARE BOSONS
- " " HALF-ODD INTEGER " " FERMIONS

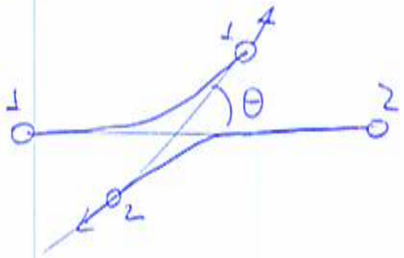
ACCORDING TO FEYNMAN, THIS THEOREM CANNOT BE UNDERSTOOD FROM SIMPLE BASIC PRINCIPLES.
THUS, IT MEANS THAT OUR UNDERSTANDING ON THE SUBJECT NEEDS TO BE IMPROVED.

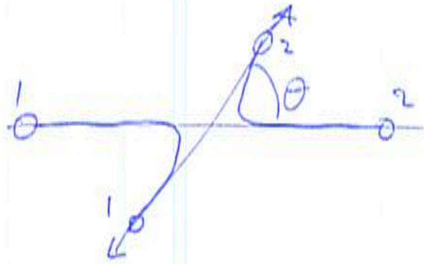
IT IS WORTHY NOTING THAT, SINCE FEYNMAN, THERE HAVE BEEN DEVELOPMENTS.

- SEE BERRY + ROBBINS, PROC. R. SOC. LONDON A, **453**, 1771 (1997)

- SEE NEW EXPERIMENTS ON DIRECT MEASUREMENT OF THE EXCHANGE PHASE, ROOS ET. AL PHYS. REV. LETT. **119**, 160401 (2017)

## CONSEQUENCES FOR THE SCATTERING EXPERIMENT




$\frac{e^{ikr}}{r}\psi(\theta,\phi)$ REPRESENTS

PARTICLE 1 $\rightarrow$ $\theta$ AND $\phi$

" 2 $\rightarrow$ $\pi-\theta$ AND $\phi+\pi$

$\frac{e^{ikr}}{r}\psi(\pi-\theta,\ \phi+\pi)$ REPRESENTS

PARTICLE 1 $\rightarrow$ $\pi-\theta$ AND $\phi+\pi$

" 2 $\rightarrow$ $\pi$ AND $\phi$

DUE TO CYLINDRICAL SYMMETRY $\psi(\theta,\phi)=\psi(\theta)$

$$\Rightarrow |\psi_{out}\rangle \propto \left(|\psi(\theta)\rangle + e^{i\gamma}|\psi(\pi-\theta)\rangle\right)\otimes\left|\frac{e^{ikr}}{r}\right\rangle$$

$\gamma=0$ BOSONS

$\gamma=\pi$ FERMIONS

PROB OF HITTING THE DETECTOR AT $\theta$ AND <u>ANY</u> $\phi$ IS

$$\sigma(\theta)\propto P(\theta) \propto |\psi_{out}|^2 \propto |\psi(\theta)|^2+|\psi(\pi-\theta)|^2+\underbrace{e^{i\gamma}\langle\psi(\theta)|\psi(\pi-\theta)\rangle + h.c.}_{\text{INTERFERENCE}}$$

FOR $\theta=\pi/2$

$$\Rightarrow P(\theta)\rightarrow \left|\psi\left(\tfrac{\pi}{2}\right)\right|^2(1+\cos\gamma)$$

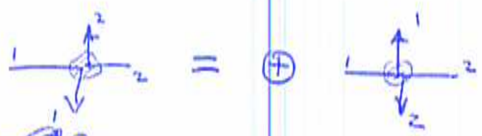
THUS, FOR $\begin{cases} \gamma = 0 \rightarrow \text{CONSTRUCTIVE INTERFERENCE} \\ \gamma = \pi \rightarrow \text{DESTRUCTIVE} \quad '' \end{cases}$

PHYSICAL INTERPRETATION


BECAUSE OF CYLINDRICAL SYMMETRY
$\phi \rightarrow \phi + \pi$

BUT THIS OPERATION IS EQUIVALENT OF EXCHANGING PART. 1 AND 2 IN THE FINAL STATE

BOSONS $(\gamma = 0)$ $\longrightarrow$




FERMIONS $(\gamma = \pi)$ $\longrightarrow$


THEREFORE, DESTRUCTIVE INTERFERENCE FOR FERMIONS

## CONSTRUCTING SYMMETRICAL AND ANTISYMMETRICAL STATES

- THE 2-PARTICLES CASE

GROUP OF PERMUTATIONS $\equiv \{ P_{12}, P_{21} \} = \{ \mathbb{1}, P_{21} \}$

$\downarrow$ OPERATOR THAT PERMUTES PARTICLES 1 AND 2

$$P_{21}\, \psi(q_1, q_2) = \psi(q_2, q_1)$$

~~SUPPOSE~~ LET $\{ \phi_i \}$ BE A SET OF $\underset{\text{ORBITAL}}{\text{1-PARTICLE}}$ WAVE FUNCTIONS WHICH IS COMPLETE AND ORTHONORMAL

WHAT ARE THE POSSIBLE SYMMETRIC AND ANTI-SYMMETRIC WAVE FUNCTIONS ?

$$|\psi_{PHYS}(\vec{r}_1, \vec{r}_2)\rangle = \text{SYMMETRIZER}\left( |\psi_\alpha(\vec{r}_1)\rangle \otimes |\psi_\beta(\vec{r}_2)\rangle \right) = S\,|\psi_\alpha, \psi_\beta\rangle$$

OR

$$\text{ANTI-SYMMETRIZER}\left( |\psi_\alpha(\vec{r}_1)\rangle \otimes |\psi_\beta(\vec{r}_2)\rangle \right) = A\,|\psi_\alpha, \psi_\beta\rangle$$

where $|\psi_\alpha\rangle = \sum_i a_{\alpha_i} |\varphi_i\rangle$

$|\psi_\beta\rangle = \sum_i a_{\beta_i} |\varphi_i\rangle$

THUS, IN THE END, WE WILL HAVE TO SYMMETRIZE OR ANTISYMMETRIZE ORTHONORMAL ORBITS. FOR SIMPLICITY, LETS CONSIDER THAT $|\psi_\alpha\rangle$ AND $|\psi_\beta\rangle$ ARE EITHER ORTHOGONAL TO EACH OTHER (OR EQUAL) TO EACH OTHER.

THE OPERATION OF SYMMETRIZATION (S) AND THE OPERATION OF ANTISYMMETRIZATION (ALSO CALLED AS ALTERNATION) (A) ARE

$$S = \frac{\mathbb{1} + P_{21}}{2} \quad , \quad A = \frac{\mathbb{1} - P_{21}}{2}$$

$$\Rightarrow \begin{cases} |\psi_{SYM}\rangle = \text{CONST} \times S |\psi_\alpha, \psi_\beta\rangle \\ |\psi_{ANTISYM}\rangle = \text{CONST} \times A |\psi_\alpha, \psi_\beta\rangle \end{cases}$$

FOR $|\psi_\alpha\rangle \neq |\psi_\beta\rangle \Rightarrow$

$$|\psi_S\rangle = \frac{1}{\sqrt{2}}\left( |\psi_\alpha \psi_\beta\rangle + |\psi_\beta \psi_\alpha\rangle \right)$$

$$|\psi_A\rangle = \frac{1}{\sqrt{2}}\left( |\psi_\alpha \psi_\beta\rangle - |\psi_\beta \psi_\alpha\rangle \right)$$

FOR $|\psi_\alpha\rangle = |\psi_\beta\rangle \Rightarrow \begin{cases} |\psi_S\rangle = |\psi_\alpha \psi_\alpha\rangle \\ |\psi_A\rangle = 0 \end{cases}$ (PAULI EXCLUSION PRINCIPLE)

HOW ABOUT THE SPINS? FOR $s = 1/2$ PARTICLES, THEN THE PHYSICAL SPINORIAL PART BECOMES

$$|\chi_S\rangle = \begin{cases} |++\rangle \\ \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) \\ |--\rangle \end{cases} \quad \text{AND} \quad |\chi_A\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

(TRIPLET) $S=1$ (SINGLET) $S=0$

THUS, FOR 2 SPIN-1/2 PARTICLES (FERMIONS), THE PHYSICAL STATES ARE

$$|\psi_{PHYS}\rangle = |\psi_S\rangle \otimes |\chi_A\rangle \quad \text{AND} \quad |\psi_A\rangle \otimes |\chi_S\rangle$$

NOTICE THAT $P_{21}|\psi_{PHYS}\rangle = -|\psi_{PHYS}\rangle$

BECAUSE $S + A = \mathbb{1}$ AND $AS = 0 = SA$, AND $S^2 = S$, $A^2 = A$

THERE ARE ONLY 2 CLASSES OF SYMMETRIC STATES.

THIS IS NO LONGER TRUE FOR MORE THAN $N=2$ PARTICLES

- 3-PARTICLES CASE

GROUP OF PERMUTATIONS $= \{ \mathbb{1}, P_{213}, P_{321}, P_{132}, P_{312}, P_{231} \}$

$\mathbb{1} = P_{123}$

$\underbrace{P_{213}: 2 \leftrightarrow 1,\ 3 \leftrightarrow 3 \qquad P_{321}: 1 \leftrightarrow 3,\ 2 \leftrightarrow 2 \qquad P_{132}: 1 \leftrightarrow 1,\ 2 \leftrightarrow 3}_{\text{1 PAIR OF PARTICLES BEING PERMUTED (SIMPLE PERMUTATION) = TRANSPOSITION}}$

$\underbrace{P_{312}: 1 \to 2 \to 3 \to 1 = P_{321}P_{213} \qquad P_{231}: 2 \to 1,\ 1 \to 3,\ 3 \to 2 = P_{213}P_{32}}_{\text{2 SIMPLE PERMUTATIONS (OR 2 TRANSPOSITIONS)}}$

NOTICE THAT THE HAMILTONIAN COMMUTES WITH ANY OPERATOR OF THE GROUP

$$P^+_{ijk} H P_{ijk} = H$$

BUT THIS IS A NON ABELIAN GROUP $P_{ijk}P_{\alpha\beta\gamma} \neq P_{\alpha\beta\gamma}P_{ijk}$

$\Rightarrow$ IMPOSSIBLE TO CONSTRUCT STATES THAT ARE EIGENSTATES OF $H$ AND OF ALL OPERATORS OF THE GROUP

## ANTI-SYMMETRIC STATE

$$|\psi_A\rangle = \text{CONST} * \sum_{\{P\}} (-1)^p \, P \, |\psi\rangle$$

FOR 3-PARTICLES, LET $|\psi\rangle = |R,G,B\rangle$ ; WITH $R \neq G$, $R \neq B$, $G \neq B$

$\{P\}$ ≡ SET OF ALL POSSIBLE PERMUTATIONS
/ THE IRREDUCIBLE REPRESENTATION OF THE GROUP

$p$ = PARITY OF THE PERMUTATION

$= \begin{cases} -1, & \text{IF THE NO. OF SIMPLE PERMUTATION IS ODD} \\ +1, & \text{" " " " " " " EVEN} \end{cases}$

AND $\text{CONST} = \frac{1}{\sqrt{3!}}$ ( FOR N FERMIONIC PARTICLES $\text{CONST} = \frac{1}{\sqrt{N!}}$ )

$$\Rightarrow \boxed{|\psi_A\rangle = \frac{1}{\sqrt{6}}\Big( |RGB\rangle + |GBR\rangle + |BRG\rangle - |GRB\rangle - |BGR\rangle - |RBG\rangle \Big)}$$

THIS IS THE BARIONIC COLOR SINGLET

## SYMMETRIC STATE

$$|\psi_S\rangle = \text{CONST} * \sum_{\{P\}} P|\psi\rangle \quad , \quad \text{FOR } R \neq G,\ R \neq B,\ B \neq G$$

$$\Rightarrow \boxed{|\psi_S\rangle = \frac{1}{\sqrt{6}}\Big( |RGB\rangle + |GBR\rangle + |BRG\rangle + |GRB\rangle + |BGR\rangle + |RBG\rangle \Big)}$$

FOR $R = G \neq B \Rightarrow |\psi_S\rangle = \sqrt{\frac{2!}{3!}} \times \frac{1}{2} \sum_{\{P\}} P|RRB\rangle = \frac{1}{\sqrt{3}}\Big(|RRB\rangle + |RBR\rangle + |BRR\rangle\Big)$

FOR $R = G = B \Rightarrow |\psi_S\rangle = \sqrt{\frac{3!}{3!}} \times \frac{1}{6} \sum_{\{P\}} P\{RRR\} = |RRR\rangle$

IN GENERAL, FOR A SYSTEM OF $N$ PARTICLES THE SYMMETRIC AND ANTI-SYMMETRIC WAVE FUNCTIONS ARE OBTAINED VIA THE FORMULAS

$$\psi_A(q_1 \cdots q_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_1(q_1) & \psi_1(q_2) & \cdots & \psi_1(q_N) \\ \psi_2(q_1) & \psi_2(q_2) & \cdots & \psi_2(q_N) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_N(q_1) & \psi_N(q_2) & \cdots & \psi_N(q_N) \end{vmatrix}$$

WHICH IS A SLATER DETERMINANT

FOR BOSONIC PARTICLES, IT IS NOT A DETERMINANT, BUT A PERMANENT. THE NORMALIZATION IS NOT SO SIMPLE DUE TO ~~REPEA~~ POSSIBLE REPEATED STATES

$$\psi_S(q_1 \cdots q_N) = \cancel{\frac{N_1 N_2 \cdots N_N}{N!}} \begin{bmatrix} \psi_1(q_1) & \cdots & \psi_1(q_N) \\ \vdots & \ddots & \vdots \\ \psi_N(q_1) & \cdots & \psi_N(q_N) \end{bmatrix} * \text{CONST}$$

WHERE $N_i$ IS THE NO. OF TIMES A STATE IS OCCUPIED BY A BOSONIC PARTICLE

AND CONST IS THE NORMALIZATION CONSTANT

$$= \sqrt{\frac{1}{N_0!\, N_1! \cdots N_N!}}$$