List of exercises #3 - 7600037

1. **Dirac-delta perturbation to a particle in a 1D box.** (Cohen-Tannoudji, Chap XI, ex. 1)
A particle of mass m is placed in an infinite one-dimensional well of width a: $V(x) = 0$ for $0 < x < a$, and $V(x) = \infty$ everywhere else. It is subject to a perturbation $W(x) = \varepsilon a \delta(x - a/2)$, where ε is a real constant with the dimensions of an energy.
   (a) Calculate, to first order in ε, the corresponding energy shifts.
   (b) Actually, the problem is exactly soluble. Setting $\hbar k = \sqrt{2mE}$, show that the possible energies are given by one of the two equations $\sin(ka/2) = 0$ or $\tan(ka/2) = -\hbar^2 k/ma\varepsilon$. Discuss the results obtained with respect to the sign and magnitude of ε. In the limit $\varepsilon \to 0$, show that one obtains the results of the preceding question.

2. **Step-potential perturbation to a particle in a 2D box.** (Cohen-Tannoudji, Chap XI, ex. 2)
Consider a particle of mass m in an infinite 2D potential well of width a: $V(x, y) = 0$ if $0 < x < a$ and $0 < y < a$, and $V(x, y) = \infty$ everywhere else. This particle is also subject to a perturbation $W(x, y) = \varepsilon$ if $0 < 2x < a$ and $0 < 2y < a$, and $W(x, y) = 0$ everywhere else, where ε is a real constant with dimension of energy.
   (a) Calculate, to first order in ε, the perturbed energy of the ground state.
   (b) Same question for the first excited state (which is degenerate). Give the corresponding wave functions to zeroth order in ε.

3. **Anysotropic and rotated 2D Harmonic Oscillator.** (Cohen-Tannoudji, Chap XI, ex. 3)
Consider the 2D isotropic Harmonic Oscillator whose Hamiltonian is $H_0 = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 R^2$ perturbed by $W = \lambda_1 W_1 + \lambda_2 W_2$ where $\lambda_{1,2}$ are dimensionless real constants, and $W_1 = m\omega^2 XY$ and $W_2 = \hbar\omega\left((L_z/\hbar)^2 - 2\right)$, where $L_z = XP_y - YP_x$ is the polar component of the orbital angular momentum of the particle.
In the following, consider only the corrections to first order for the energies and to zeroth order for the state vectors.
   (a) Indicate without calculations the eigenvalues of H_0, their degrees of degeneracy and the associated eigenvectors.
   In what follows, consider only the second excited state of H_0, of energy $3\hbar\omega$ and which is three-fold degenerate.
   (b) Calculate the matrices representing $W_{1,2}$.
   (c) Assume $\lambda_2 = 0$ and $\lambda_1 \ll 1$. Calculate, using perturbation theory, the effect of $\lambda_1 W_1$, and compare the results with the limited expansion of the exact solution.
   (d) Assume $\lambda_2 \ll \lambda_1 \ll 1$. Calculate, using perturbation theory, the effect of $\lambda_2 W_2$.
   (e) Assume $\lambda_1 = 0$ and $\lambda_2 \ll 1$. Calculate, using perturbation theory, the effect of $\lambda_2 W_2$, and compare the results with the limited expansion of the exact solution.
   (f) Finally, assume that $\lambda_1 \ll \lambda_2 \ll 1$. Calculate, using perturbation theory, the effect of $\lambda_1 W_1$.

4. **Zeeman effect on an easy-axis and easy-plane spin.** (Cohen-Tannoudji, Chap XI, ex. 5)
Consider a system of total angular momentum equal to $\hbar$, i.e., $\ell = 1$. The corresponding kets are $\{|m\rangle\}$, with $m = -1, 0, 1$. The unperturbed system Hamiltonian is $H_0 = aJ_z + bJ_z^2/\hbar$, where a and b are two positive constants, which have the dimensions of an angular frequency.
   (a) What are the energy levels of the system? For what value of the ratio b/a is there degeneracy?
   (b) A static field $\mathbf{B} = B(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ is applied. The corresponding perturbation to H_0 is $W = -g\mathbf{B}\cdot\mathbf{J} = \omega\hat{B}\cdot\mathbf{J}$, where g is the gyromagnetic ratio, assumed to be negative, and $\omega = -gB$ is the corresponding Larmor angular frequency. Write the matrix which represents W in the basis of the three eigenstates of H_0.
   (c) Assume that $b = a \gg \omega$ and that $\mathbf{B} = B\hat{x}$. Calculate the energies and eigenstates of the system, to first order in ω for the energies and to zeroth order for the eigenstates.
   (d) Assume that $b = 2a \gg \omega$ (but the direction of $\mathbf{B}$ is arbitrary). What is the expansion of the ground state of $H_0 + W$ to first order in ω?

(e) Calculate the mean value $\langle \mathbf{M} \rangle$ of the magnetic moment $\mathbf{M} = g\mathbf{J}$ of the system in that ground state. Are $\langle \mathbf{M} \rangle$ and $\mathbf{B}$ parallel? Show that one can write $\langle M_i \rangle = \sum_j \chi_{ij} B_j$, with $i, j = x, y, z$. Calculate the coefficients $\chi_{i,j}$ (the components of the susceptibility tensor).

5. **Anharmonic oscillator.**
Consider a 1D Harmonic Oscillator perturbed by a quartic term, i.e., $H_0 = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2$ and $V = \lambda\hbar\omega\left(\frac{x}{a}\right)^4$, where $a = \sqrt{\frac{\hbar}{m\omega}}$ and $0 < \lambda \ll 1$ is a small constant. Treating V as a perturbation to H_0, obtain the energy corrections up to second order and the state corrections up to first order.

6. **The K-electron energy shift in a heavy nucleus.**
The "radius" of a proton is roughly $r_p = 10^{-15}$ m. The "orbit" of the $n = 1$ state in Hydrogen is roughly $a_0 = \frac{\hbar^2}{me^2} \approx 0.5$ Å $\sim 10^5 r_p$. Hence, it is an excellent approximation to assume a point proton.
In a heavy nucleus atoms, however, (say, $Z \sim 80$), the electron orbit diminishes $a_0 \to a = a_0/Z \sim 10^3 r_p$ and the nucleus radius $R \sim 10 r_p$, so that the assumption of a point-like nucleus is no longer valid. There is therefore a shift in the $n = 1$ energy level due to the finite nuclear size, which can be estimated by first-order perturbation theory.
For the purposes of this calculation, assume that a single electron interacts with a nuclear charge $Q = Ze$ which is uniformly distributed throughout a structureless sphere of radius R. Also, assume that the nuclear mass is much greater than the electron's.

(a) Use classical electrostatics to determine the perturbation $V(Z, R, \mathbf{r})$ where $\mathbf{r}$ is the electron's position coordinate.

(b) The integral yielding the first-order correction to the nth energy is trivial to calculate if an approximation, based on the facts that $R \approx 7r_p$ and $Z \approx 80$, is made. Find this approximation, show why it is valid, and then use it to evaluate the energy correction. Does it depend on the total angular momentum (ℓ) or on its z component (m)?

(c) Now set $R = 7r_p$ and $Z = 81$ and calculate the numerical values of the unperturbed energy and the corresponding correction for $n = 1$. Discuss whether first-order perturbation theory is reliable for this situation.

7. Develop the second-order time-independent perturbation theory when the degeneracy is not removed in the first order.

8. The system Hamiltonian, in dimensionless units, is $H = H_0 + \lambda V$, where

$$H_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

with $|\lambda| \ll 1$ being a constant.

(a) Compute the energy correction to all states up to the first nonvanishing order in perturbation theory and compare with the exact result.

(b) Compute the corresponding eigenstates up to zeroth order in perturbation theory and compare with the exact result.