List of exercises #2 - 7600037

1. Cohen-Tannoudji, exercises 2 and 3 from chapter IX (complement B).
2. Cohen-Tannoudji, exercises 1 to 5 (except 2 which is optional) from chapter X (complement G).
3. (Optional) The Wigner-Eckart theorem. Cohen-Tannoudji, exercise 8 from chapter X (complement G).
4. Cohen-Tannoudji, complement A from chapter IX.
   (a) Show that the operator $e^{-i\boldsymbol{\sigma}\cdot\hat{n}\phi/2} = \mathbb{1}\cos\left(\frac{\phi}{2}\right) - i\boldsymbol{\sigma}\cdot\hat{n}\sin\left(\frac{\phi}{2}\right)$, where $\boldsymbol{\sigma}$ are the Pauli matrices, $\hat{n}$ is a unit vector, and $\phi \in \mathbb{R}$.
   (b) What is the physical interpretation of the operator in the previous item? What is the significance for $\phi = 2\pi$?
5. (Optional) Cohen-Tannoudji, complement D from chapter X.
6. Let $\mathbf{J}$ be the total angular momentum of a physical system.
   (a) Compute $\mathbf{J}' = e^{-\frac{i}{\hbar}\phi J_z}\mathbf{J}e^{\frac{i}{\hbar}\phi J_z}$. (*Hint*: Use the Baker-Haussdorff identity: $e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \ldots$.)
   (b) Interpret geometrically your result for $\mathbf{J}'$.
   (c) Based on this interpretation, what would be the result for $\mathbf{J}'' = e^{-\frac{i}{\hbar}\phi\hat{n}\cdot\mathbf{J}}\mathbf{J}e^{\frac{i}{\hbar}\phi\hat{n}\cdot\mathbf{J}}$, where $\hat{n}$ is a unitary vector?
   (d) And for $\mathbf{J}''' = e^{-\frac{i}{\hbar}\theta J_x}e^{-\frac{i}{\hbar}\phi J_z}\mathbf{J}e^{\frac{i}{\hbar}\phi J_z}e^{\frac{i}{\hbar}\theta J_x}$?
7. Show the identity used in class: $e^{-\frac{i}{\hbar}\mathbf{d}\cdot\mathbf{P}}\mathbf{R}e^{\frac{i}{\hbar}\mathbf{d}\cdot\mathbf{P}} = \mathbf{R} - \mathbf{d}\mathbb{1}$, where $\mathbf{d}$ is a vector, and $\mathbf{R}$ and $\mathbf{P}$ are the usual position and momentum operators, respectively.
8. Cohen-Tannoudji, complement F from chapter II.
   (a) How does the momentum operator ($\mathbf{p}$) transform under parity ($\boldsymbol{\pi}$)?
   (b) A quantum mechanical state $|\psi\rangle$ is known to be Eigenstate of momentum and parity simultaneously. What can be said about the eigenvalues?
   (c) Give a physical interpretation of your results in the previous item.
9. (Optional) Cohen-Tannoudji, complement B from chapter VII; and Shankar, Sec. 15.4.
   (a) Show that the Runge-Lenz vector $\mathbf{N} = \mathbf{p}\times\mathbf{L} + \alpha\hat{r}$ is a conserved quantity for a central force $\mathbf{F} = F(r)\,\hat{r}$, with $F(r) \propto r^{-2}$, and α is a constant to be determined. What is the value of α for the gravitational force?
   (b) Using that the orbital angular momentum $\mathbf{L}$ and $\mathbf{N}$ are conserved, show the first law of Kepler.
   (c) In quantum mechanics, it is useful to use the symmetrization rule $O \to \frac{1}{2}\left(O + O^\dagger\right)$. Using this rule, show that the Runge-Lenz vector operator is $\mathbf{N} = \frac{1}{2}(\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}) + \alpha\hat{r}$.
   (d) Show that $\mathbf{N}$ commutes with the Hydrogen atom Hamiltonian and compute α. This additional symmetry is responsible for the larger degeneracy in the Hydrogen atom when compared with the 3D Harmonic Oscillator.
   (e) What is the implication for the wavefunction $\Psi_{1,0,0}$ in exercise 7(g) of List #1?
10. Consider a particle subjected to the action of a periodic external potential $V(x)$ of period $a \neq 0$, i.e., $V(x+a) = V(x)$ and $V(x+b) \neq V(x)$ for b different from a multiple of a. [For example, $V(x) = V_0\sin(2\pi x/a)$.]
    (a) Is the momentum of the particle conserved? Justify.
    (b) Let $\tau(\ell)$ be the spatial translation operator defined by $\tau^\dagger(\ell)\,x\tau(\ell) = x + \ell\mathbb{1}$. Write down the operator $\tau(\ell)$ explicitly.
    Hint: You do not need to derive $\tau(\ell)$, just write it down and show that $\tau^\dagger(\ell)\,x\tau(\ell) = x + \ell\mathbb{1}$.
    (c) Show that $\tau(\ell)\,|x\rangle = |x+\ell\rangle$, besides of a phase.

(d) For what values of ℓ does the system become invariant under spatial translations?

(e) Let the ket $|n\rangle$ be the state of a particle confined in the nth valley of $V(x)$. (For example, $\langle x|n\rangle$ is a Gaussian of width much smaller than a and centered at the nth valley.) Show that

$$|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{in\theta}\,|n\rangle$$

is an eigenstate of $\tau(a)$, and compute the corresponding eigenvalue.

(f) Show that the eigenfunction $\theta(x) = \langle x|\theta\rangle$ can be written as the combination of a plane wave multiplied by a function that has the same period as the external potential $V(x)$, that is, $\theta(x) = e^{i\theta x/a}u_\theta(x)$, with $u_\theta(x+a) = u_\theta(x)$.
Hint: Use the results from parts (c) and (e), and that $\theta(x) = \langle x|\theta\rangle = \left\langle x\left|\tau^\dagger(a)\,\tau(a)\right|\theta\right\rangle$.
[Note that, setting $k = \theta/a$, we have that $\theta(x) = \psi_k(x) = e^{ikx}u_k(x)$, i.e., the eigenfunction is a periodic function times a plane wave. This is Bloch's theorem for the one-dimensional case.]

(g) Is the state $|\theta\rangle$ an eigenstate of the Hamiltonian? Justify.