List of exercises #1 - 7600037

1. Let $\mathbf{J}$ be an angular momentum operator in quantum mechanics, is the relation $\mathbf{J} \times \mathbf{J} = i\hbar \mathbf{J}$ true of false? Prove it.

2. Show that the matrix element $\langle j', m' |\mathbf{J}| j, m\rangle$ is vanishing when $j' \neq j$. In addition, show that it is also vanishing for $|m - m'| > 1$. Finally, write the matrix for J_+.

3. In class, it was argued that the angular momentum operator has a discrete spectrum. Using the choice $J^2 |j, m\rangle = \hbar^2 j (j+1) |j, m\rangle$ and $J_z |j, m\rangle = \hbar m |j, m\rangle$, we have argued that j and m are integers or half-integers with $j \geq 0$, $m = -j, -j+1, \ldots, j-1, j$, implying that there are $2j+1$ different values of m for each j. It was then concluded that the quantum of angular momentum is $\hbar$.

   (a) Reproduce in your own way the arguments presented in class for these conclusions. (Naturally, you can present an alternative way.)

   (b) If we had used the choice $\left[J^\alpha, J^\beta\right] = i\hbar g \sum_\gamma \epsilon_{\alpha\beta\gamma} J^\gamma$, with $g \in \Re_+^*$, what would be the values of m and j? What about the quantum of angular momentum? Why $g = 1$ is the value used in the literature? Is there a fundamental reason for that?

4. In mathematics, the special unitary group of degree N, denoted by SU(N), is the Lie group of $N \times N$ unitary matrices with determinant 1. This means that any element of the group can be written as e^{iT}, with T being a traceless hermitean $N \times N$ matrix.

   (a) For the SU(2) group, show that the matrices

   $$T_1 = \frac{1}{2}\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; T_2 = \frac{1}{2}\sigma_y = \frac{i}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T_3 = \frac{1}{2}\sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   are a basis for any traceless hermitean 2×2 matrix.

   (b) Show that the matrices T_α obey the angular momentum algebra

   $$[T_\alpha, T_\beta] = i \sum_\gamma \epsilon_{\alpha,\beta,\gamma} T_\gamma.$$

   It is then said that these matrices define what is called the (2-dimensional) spinor representation of the SU(2) algebra.

   (c) Show that any T matrix has discrete spectrum and that the Eigenvalues are simply $\pm\alpha$, with $\alpha \in \Re$. What is the value of α?

   (d) Compute the corresponding Eigenvectors in the T_3-basis. What is the corresponding Eigenvalues of $T^2 = T_1^2 + T_2^2 + T_3^2$? (Any higher dimensional matrix can be constructed from the T_α matrices implying that their spectrum are quantized just as in exercise 3. Thus, the quantization of angular momentum is not due to $\hbar$, but due to the algebra. In addition, the matrix T^2 is known as the Casimir and is proportional to the identity.)

5. Consider a generic two-level system and define the operator $\rho = |\psi\rangle\langle\psi|$ (known as density matrix), where $|\psi\rangle$ is a generic ket state of the system.

   (a) Show that $\mathrm{tr}(\rho) = 1$.

   (b) Show that ρ can be expanded as

   $$\rho = \sum_{i=0}^{3} a_i \sigma_i,$$

   where σ_0 is the identity matrix, $\sigma_{1,2,3}$ are the Pauli matrices, and $a_i \in \Re$. Determine the coefficients a_i.

6. (Cohen-Tannoudji, exercise VII.1) Let ρ, θ and z be the cylindrical coordinates of a spinless particle on an external potential $U \equiv U(\rho)$.

(a) Write in cylindrical coordinates the differential operator associated with the Hamiltonian.

(b) Show that H commutes with L_z and P_z. Show from this that the wavefunction associated with the stationary states of the particle can be chosen in the form

$$\Psi_{n,m,k}(\rho,\theta,z) = f_{n,m}(\rho)\, e^{im\theta} e^{ikz},$$

where the values of m and k are to be specified.

(c) Write, in cylindrical coordinates, the Eigenvalue equation for H and derive the corresponding differential equation for $f_{n,m}$.

(d) Let the $\mathcal{R}_y$ be the operator whose action, in the $\{|\mathbf{r}\rangle\}$ representation, is to change y to $-y$ (reflection with respect to the xz plane.) Does $\mathcal{R}_y$ commute with H?

(e) Show that $\mathcal{R}_y$ anticommutes with L_z, and thus $\mathcal{R}_y\,|\Psi_{n,m,k}\rangle$ is an Eigenvector of L_z. What is the corresponding Eigenvalue? What can be concluded concerning the degeneracy of the energy levels of the particle? Could this result be predicted directly from the differential equation established for H?

7. Consider the single-particle Hamiltonian

$$H = \frac{1}{2m}\left(P_x^2 + P_y^2 + P_z^2\right) + \frac{1}{2}m\omega^2\left(X^2 + Y^2 + Z^2\right) = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 R^2,$$

and the vector operator

$$\mathbf{T} = \sqrt{\frac{m\omega}{2\hbar}}\left(\mathbf{R} - \frac{i}{m\omega}\mathbf{P}\right).$$

(a) Show that $H = \hbar\omega\left(\mathbf{T}\cdot\mathbf{T}^\dagger + \frac{3}{2}\right)$. (In view of this, T_α can be viewed as a creation operator in direction $\hat{\alpha}$.)

(b) Show that $[H, T_\alpha] = \hbar\omega T_\alpha$.

(c) Why the Eigenfunctions of H can be chosen in the form $\Psi_{n,\ell,m}$ which are simultaneously Eigenfunctions of H, L^2 and L_z such that $H\,|\Psi_{n,\ell,m}\rangle = \left(n + \frac{3}{2}\right)\hbar\omega\,|\Psi_{n,\ell,m}\rangle$?

(d) Assuming that the ground state is nondegenerate, use the result to the 1D harmonic oscillator in order to compute $\Psi_{0,0,0}$.

(e) Let $T_+ = T_x + iT_y$. Show that

i. $[H, T_+] = \hbar\omega T_+$,

ii. $[L_z, T_+] = \hbar T_+$,

iii. $\left[L^2, T_+\right] = \hbar\left(\{L_z, T_+\} - \{L_+, T_z\}\right)$, and

iv. $\left[\mathbf{L}, T^2\right] = 0$.

(f) Using those commutators, show that

i. $T_+ Y_{\ell,\ell} \propto Y_{\ell+1,\ell+1}$,

ii. $T_+ \Psi_{n,\ell,\ell} \propto \Psi_{n+1,\ell+1,\ell+1}$,

iii. $(T_+)^n\, \Psi_{0,0,0} \propto \Psi_{n,n,n}$, and

iv. $\left(T^2\right)^n \Psi_{0,0,0} \propto \Psi_{2n,0,0}$.

(g) Finally, show that $\Psi_{n,\ell,m} \propto (L_-)^{\ell-m}\,(T_+)^\ell\left(T^2\right)^{\frac{n-\ell}{2}}\Psi_{0,0,0}$. Compute and normalize $\Psi_{1,0,0}$ and $\Psi_{1,1,0}$.

8. One of the relativistic corrections to the Hydrogen atom (given by the unperturbed Hamiltonian H_0) is called the spin-orbit term

$$\delta H = \frac{1}{2m^2c^2}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}U(r)\right)\mathbf{L}\cdot\mathbf{S},$$

where m is the electron mass, U is the Coulombian interaction energy, and $\mathbf{L}$ and $\mathbf{S}$ are, respectively, the electron orbital and spin angular momentum.

(a) What is the physical origin of δH? Can you deduce it (apart from the $\frac{1}{2}$ factor)?

(b) Show (for the grond-state, for instance) that $\delta H/E_0 \propto \alpha^2$, where α is the fine-structure constant.

(c) Show that the the $[H_0 + \delta H, \mathbf{J}] = 0$, where $\mathbf{J} = \mathbf{L} + \mathbf{S}$ is the total angular momentum.

(d) What is the consequence for the degeneracy of the Eigenstates of H? Explicit for the $n = 1$ and $n = 2$ states. (Disregard the proton spin.)

(e) Notice it suggests a modification of the spectroscopic notation. For instance, $n(^{2S+1}L_J)$ means the principal number is n, L is the orbital angular momentum, S is the total spin angular momentum, which is relevant only for more than one-electron atoms, and J is the total angular momentum. For the Hydrogen atom, what means $2p_{\frac{3}{2}}$, $2p_{\frac{1}{2}}$ and $2s_{\frac{1}{2}}$? What is the degeneracy of these states in the presence of spin-orbit coupling? What is the corresponding energy difference?

9. (Sakurai, chap. 3) The wave function of a particle subjected to a spherically symmetric potential $V(r)$ is

$$\Psi(\mathbf{r}) = f(r)\,(x + y + 3z)\,.$$

(a) Is $\Psi(\mathbf{r})$ an Eigenfunction of $\mathbf{L}^2$? If so, what is the corresponding ℓ? If not, what are the possible values of ℓ and their corresponding probabilities if one measures $\mathbf{L}^2$?

(b) What are the probabilities for the particle to be found in various m_j sates?

(c) Suppose it is known somehow that $\Psi(\mathbf{r})$ is an energy Eigenfunction with energy E. Indicate how we may find $V(r)$.

10. Let $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3$, where $\mathbf{S}_i$ (with $i = 1, 2, 3$) are spin-1/2 operators.

(a) What are the eigenvalues and degeneracies of S^2?

(b) What are the corresponding eigenstates?

(c) Are S^2 and S_z a complete set of commuting observables? In other words, do they form a complet set whose eigenvectors form a basis to any quantum state of the 3 operator $\mathbf{S}_i$? Justify. If not, which operator (or operators) can be added to S^2 and S_z in order to obtain a complete set of commuting observables?

11. The state of a spin-1/2 particle is $|\psi\rangle = |\psi_+\rangle \otimes |+\rangle + |\psi_-\rangle \otimes |-\rangle$ where $\psi_\pm(\mathbf{r}) = \langle \mathbf{r}, \pm|\psi\rangle$, and $\{|\mathbf{r}, \pm\rangle\} = \{|\mathbf{r}\rangle \otimes |\pm\rangle\}$ is the position and S^z (the particle spin z component) basis. Let

$$\psi_+(\mathbf{r}) = R(r)\left[Y_{1,1}(\theta,\phi) + \frac{1}{\sqrt{3}}Y_{2,1}(\theta,\phi)\right] \text{ e } \psi_-(\mathbf{r}) = \frac{R(r)}{\sqrt{3}}\left[Y_{1,1}(\theta,\phi) - \frac{1}{\sqrt{3}}Y_{1,0}(\theta,\phi)\right],$$

where $R(r)$ is a real function and $Y_{\ell,m}(\theta,\phi)$ are the spherical harmonics.

(a) What is the normalization of $R(r)$?

(b) What is the probability of measuring L^z (the particle orbital angular momentum z component) equal to $\hbar$? Suppose the measurement was performed and the result was $\hbar$. What is the wave function right after the measurement?

(c) Right after that measurement, S^z is measured. What is the probability of obtaining $\hbar/2$?