Ensemble of topological defects and the confining flux tube

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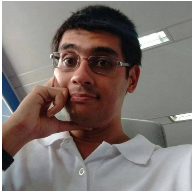

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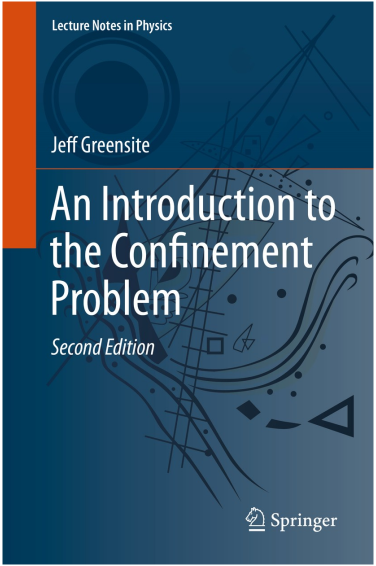
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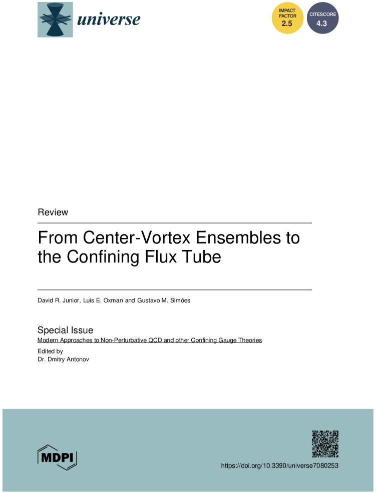
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Review
From Center-Vortex Ensembles to the Confining Flux Tube
David R. Junior, Luis E. Oxman and Gustavo M. Simões
Special Issue
Modern Approaches to Non-Perturbative QCD and other Confining Gauge Theories
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Electromagnetism and QED

Classical

$$S_{\text{Maxwell}} = \int d^4x \, \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\mathbb{F} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \, A_\mu = (\phi, \vec{A}).$$

Gauge transformation: $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha(t, \vec{x})$

Abelian group $U(1)$: $\frac{1}{e}\partial_\mu \alpha = \frac{i}{e} U \partial_\mu U^{-1}$, where $U = e^{i\alpha}$

Quantum

$$\langle \hat{\mathrm{O}} \rangle = \frac{\int [DA] O(A) e^{-S_{\text{Maxwell}}[A]}}{\int [DA] e^{-S_{\text{Maxwell}}[A]}}$$

Fermions $\psi(x)$ with $S_{\text{matter}} = \int d^4x \bar{\psi}(\not{D} + m)\psi$

$$\langle \hat{\mathrm{O}} \rangle = \frac{\int [DA][D\psi][D\bar{\psi}] O(A\,\psi,\bar{\psi}) e^{-S_{\text{QED}}[A,\psi,\bar{\psi}]}}{\int [DA][D\psi][D\bar{\psi}] e^{-S_{\text{QED}}[A,\psi,\bar{\psi}]}}$$

Gauge transformation $\psi \to U\psi$, $\bar{\psi} \to \bar{\psi} U^{-1}$

Yang-Mills theory and QCD

Non-Abelian Gauge group $SU(3)$: $SS^\dagger = \mathbb{I}$ and $\det S = 1$

$$A_\mu = \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} \text{ with } \mathrm{Tr}\, A_\mu = 0 \text{ and } A_\mu^\dagger = A_\mu$$

Gauge transformation: $A_\mu \to S A_\mu S^{-1} + \frac{i}{g} S \partial_\mu S^{-1}$ and $\psi \to S\psi$, $\bar{\psi} \to \bar{\psi} S^{-1}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\,[A_\mu, A_\nu]$$

$$S_{\mathrm{YM}} = \frac{1}{4} \int d^4x\, \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu},\ S_{\mathrm{matter}} = \int d^4x\, \mathrm{Tr}\, \bar{\psi}(\not{D} + m)\psi$$

$$\langle \hat{\mathrm{O}} \rangle = \frac{\int [DA][D\psi][D\bar{\psi}] O(A\,\psi,\bar{\psi}) e^{-S_{\mathrm{QCD}}[A,\psi,\bar{\psi}]}}{\int [DA][D\psi][D\bar{\psi}] e^{-S_{\mathrm{QCD}}[A,\psi,\bar{\psi}]}}$$

Can also be generalized to $SU(N)$ and confinement remains

Wilson Loop

- $\mathcal{W}_{\mathrm{D}}(\mathcal{C}_{\mathrm{e}}) = \frac{1}{\mathscr{D}} \operatorname{tr} \mathrm{D} \left(P \left\{ e^{i \int_{\mathcal{C}_{\mathrm{e}}} dx_\mu \, A_\mu(x)} \right\} \right)$
- In the heavy-quark limit, $\langle \mathcal{W}_{\mathrm{D}}(\mathcal{C}_{\mathrm{e}}) \rangle \sim e^{-T \, V_{\mathrm{D}}(R)}$
- Lattice: $V_D(R) = \sigma_D R + \frac{\gamma}{R} + O(1/R^2)$
- In general, confinement leads to an Area Law
- Lüscher Term: $\gamma = -\frac{\pi}{12}(D-2)$

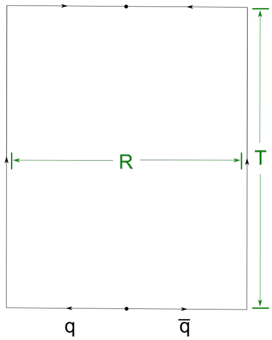


Figure: Rectangular Wilson Loop

The role of center vortices

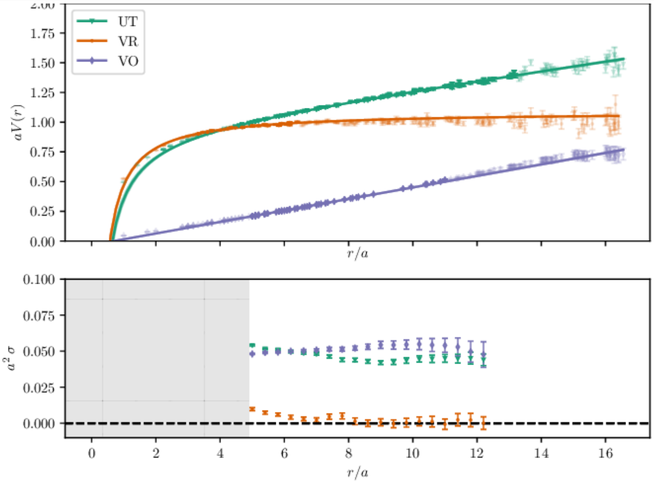
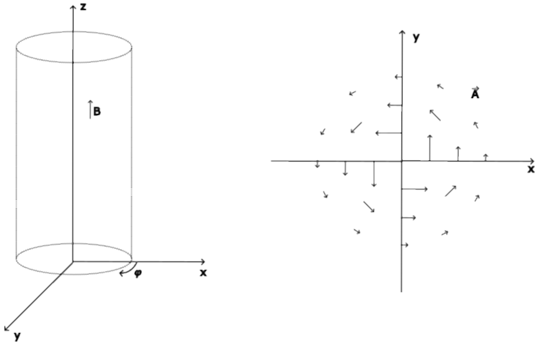


Figure: Static quark-antiquark potential for the untouched, vortex-removed and vortex-only lattice simulations. From Biddle, Kamleh, and Leinweber (2022).

Simplest example

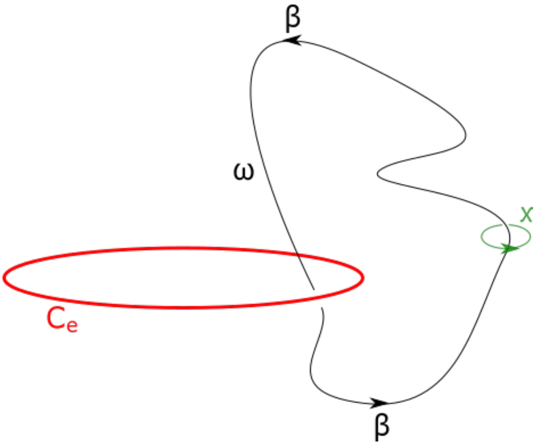
- $A_\mu = \frac{1}{g}\partial_\mu \varphi \beta \cdot T$
- $\vec{B} = \frac{1}{g}\nabla \times (\nabla\varphi)\beta \cdot T = \frac{\hat{z}}{g}\delta^{(2)}(x,y)\beta \cdot T$
- $\partial_\mu \varphi$ gives the vorticity while $\beta \cdot T$ gives the center...icity (???)
- Can be created with a singular-valued gauge transformation: $A_\mu = \frac{i}{g}S\partial_\mu S^{-1}$, $S = e^{i\varphi\beta\cdot T}$
- $\varphi \to \chi$ changes the vortex core location
- $\frac{1}{g} \to \frac{a(\rho)}{g}$ gives thickness to the vortex
- Non-Abelian d.o.f.



Straight vortex around the z axis.

Center vortex and Wilson loop

- $\mathcal{W}_{\mathcal{C}_e}[A_\mu] = \frac{1}{N}\,\mathrm{tr}\left(P\left\{e^{i\int_{\mathcal{C}_e} dx_\mu\, A_\mu(x)}\right\}\right)$
- $\langle \mathcal{W}_{\mathcal{C}_e}\rangle = \int [DA]\,\mathcal{W}_{\mathcal{C}_e}[A_\mu]\,e^{-S_{YM}}$
- Center vortices are worldsheets/worldlines in 4D/3D
- $\mathcal{W}_{\mathcal{C}_e} = \left(e^{i\frac{2\pi}{N}}\right)^{\mathrm{L}(\omega,\mathcal{C}_e)}$
- $Z(N) = \{e^{i\frac{2k\pi}{N}}\,|k = 0, 1, \ldots, N-1\}$



Center vortex line linking a Wilson loop.

Roots and weights

- Weights are $(N-1)-$vectors whose components are the eigenvalues of the diagonal generators T_q of $SU(N)$.
- Roots are $(N-1)-$vectors defined via the commutation relations of the $su(N)$ (also as the eigenvalues of $\mathrm{Ad}(T_q)$).
- For $SU(N)$, $\alpha_{ij} = \omega_i - \omega_j$.
- $\beta_1 + \beta_2 + ... + \beta_N = 0$.
- $A_\mu = \frac{1}{g}\partial_\mu \varphi \beta \cdot T \implies W_C[A_\mu] = e^{i\frac{2\pi}{N}}$.
- $A_\mu = \frac{1}{g}\partial_\mu \varphi \alpha \cdot T \implies W_C[A_\mu] = 1$.

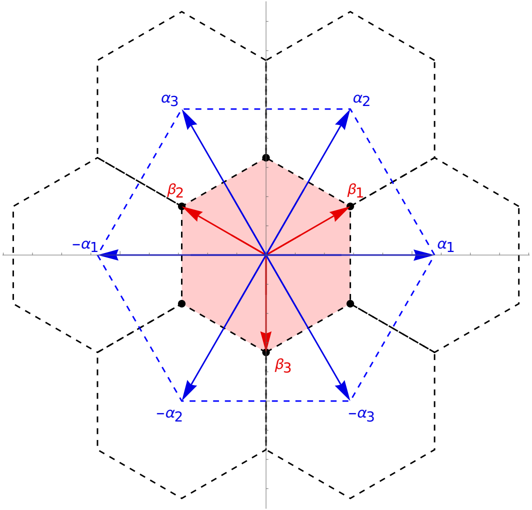
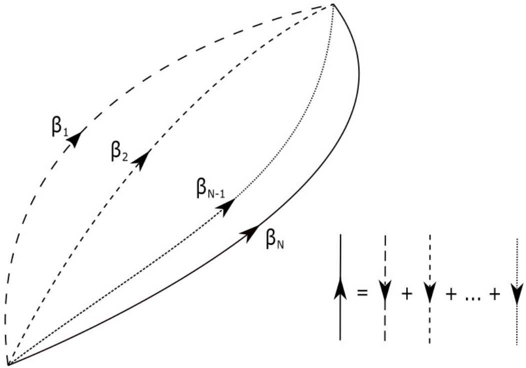


Figure: Roots and Weights of $SU(3)$.

N-lines correlations

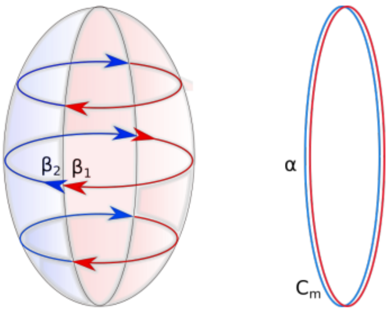
- Can be regarded as $(N-1)$ loops, although with different probability.
- $A_\mu = \sum_{i=1}^{N-1} \beta_i \cdot T\, \partial_\mu \chi_i$
- $W_\mathcal{C}[A_\mu] = \left(e^{i\,2\pi\, \beta_1 \cdot w_e}\right)^{L_1} \ldots \left(e^{i\,2\pi\, \beta_{N-1} \cdot w_e}\right)^{L_N}$
- Only works because $\beta_1 + \beta_2 + ... + \beta_N = 0$



N-vortex matching configuration. From Júnior, Oxman, and Simões (2019)

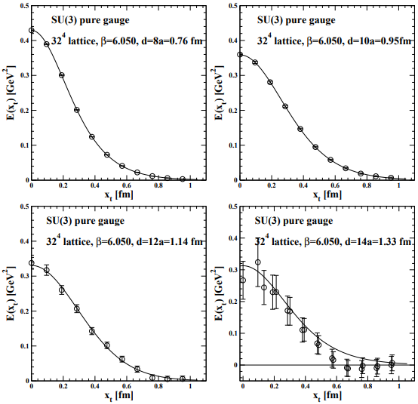
Monopoles

- Monopoles are worldlines/instantons in 4D/3D
- Cannot exist isolated. It must be either attached to a Dirac string or something else.
- In $SU(N)$, something else could be center vortices, carrying weights β_1, β_2.
- In this case, the monopole carries a root $\alpha = \beta_1 - \beta_2$ as its charge
- Can also be created with a singular-valued gauge transformation:
  $A_\mu = \frac{i}{g} S \partial_\mu S^{-1}$, $S = e^{i\varphi\beta_1 \cdot T} e^{i\sqrt{N}\theta T_\alpha}$
- Notice that $S(\varphi, \theta = 0) = e^{i\varphi\beta_1 \cdot T}$ and $S(\varphi, \theta = \pi) = e^{i\varphi\beta_2 \cdot T}$

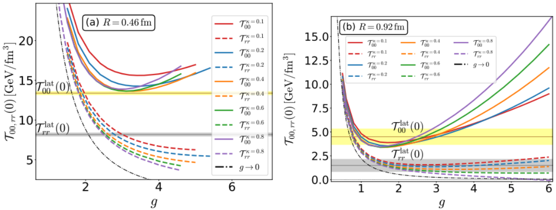


Non-oriented vortex with a monopole in the middle. From Oxman (2018)

Confinement phenomenology



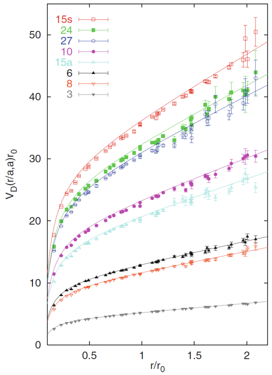
Transverse profiles of the confining string. From Cosmai et al. (2020)



Energy-momentum tensor of the confining string. From M. Kitazawa and R. Yanagihara (2019)

Scaling Laws

- Fundamental quarks: $\psi \to S\psi$
- Higher irrep. quarks: $\psi \to D(S)\psi$
- Intermidiate distances: $\frac{\sigma_{\mathrm{I}}(\mathrm{D})}{\sigma_{\mathrm{I}}(\mathrm{F})} = \frac{C_2(\mathrm{D})}{C_2(\mathrm{F})}$
- Gluonic screening for large r
- N-ality: $\mathrm{D}\left(e^{i\frac{2\pi}{N}}I\right) = \left(e^{i\frac{2\pi}{N}}\right)^k \mathbb{I}_{\mathscr{D}}$
- k-Antisymmetric irrep has the smallest quadratic Casimir.
- Large r in 3D: $\sigma_k^{(3)} = \frac{k(N-k)}{N-1} = \frac{C_2(k\text{-A})}{C_2(\mathrm{F})}$
- Large r in 4D: $\sigma_k^{(4)} = \frac{k(N-k)}{N-1}$ or $\sigma_k^{(4)} = \frac{\sin k\pi/N}{\sin \pi/N}$



Casimir scaling for $SU(3)$, from Bali (2000). The length $r_0 \approx 0.5 fm$

General Strategy

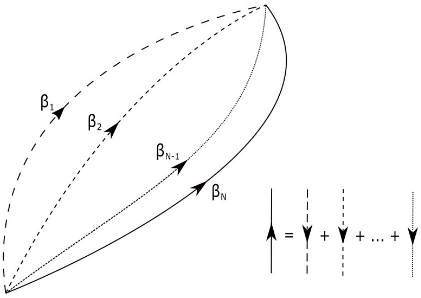
Yang Mills theory $Z_{YM} = \int [DA]\, e^{-S_{YM}}$	$\rightarrow$	Sectorization $Z_{YM} = \sum_{S_0} Z_{YM}^{(S_0)}$	$\rightarrow$	???
				$\downarrow$
Observables $\sigma_k,\ T_{\mu\nu}$	$\leftarrow$	Effective Model $S_{YMH}(\Lambda_\mu, \psi_A)$	$\leftarrow$	Ensemble $\langle W(\mathcal{C})\rangle = \sum_{\omega} e^{-S(\omega)} z_N^L(\omega, \mathcal{C})$

Worldline ensenble

- Polymer Techniques
- $Z_{\text{loops}}[b_\mu] = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{k=1}^{n} \int_0^\infty \frac{dL_k}{L_k} \int dv_k \int [dx^{(k)}]_{v_k,v_k}^{L_k} e^{-\int_0^{L_k} ds_k \left[\frac{1}{2\kappa} \dot{u}_\mu^{(k)} \dot{u}_\mu^{(k)} + \mu\right]} W_{l_k}[b_\mu]$
- μ is the string mass density and κ is the stiffness. The percolating regime is realized when $\mu < 0$ and $\kappa > 0$.
- It can be shown that $Z_{\text{loops}}[b_\mu] \approx (\det\, O)^{-1} = \int [d\phi]\, e^{-\int d^3x\, \phi^\dagger O \phi}$, where $O = -\frac{1}{3\kappa}(\partial_\mu - ib_\mu)^2 + \mu I_N$
- A Contact interaction can be included in order to generate a ϕ^4 term.
- Extended models can include more exotic interactions

N-lines correlation

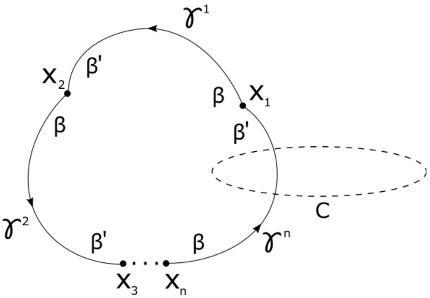
- $A_\mu = \sum_{i=1}^{N-1} \beta_i \cdot T\, \partial_\mu \chi_i$
- $W_{\mathcal{C}}[A_\mu] = \left(e^{i\,2\pi\,\beta_1 \cdot w_e}\right)^{L(S(\mathcal{C}),l_1)} \ldots \left(e^{i\,2\pi\,\beta_{N-1} \cdot w_e}\right)^{L(S(\mathcal{C}),l_{(N-1)})}$
- N possible charges lead to N fields in the effective model, which can be arranged in a $N \times N$ matrix Φ
- $\mathcal{L} = \frac{1}{3\kappa} \mathrm{Tr}((D_\mu \Phi)^\dagger D_\mu \Phi) + \mu\, \mathrm{Tr}(\Phi^\dagger \Phi) - \xi_0(\det \Phi + \det \Phi^\dagger)$



An N center-vortex creation-annihilation process.

Chains of vortices and instantons

- $S = e^{i\chi\,\beta\cdot T}\,W(x)\;,\;A_\mu = \frac{i}{g}S\partial_\mu S^{-1}$
- Ex.: $S(\varphi,\theta) = e^{i\varphi\,\beta\cdot T}\,e^{i\theta\sqrt{N}\alpha\cdot T}$
- $S(\varphi,0) = e^{i\varphi\,\beta\cdot T}$ e $S(\varphi,2\pi) \propto e^{i\varphi\,\beta'\cdot T}$
- $V_{\text{inst}} \propto \mathrm{Tr}(\Phi^\dagger T_A \Phi T_A)$
- $S_{\text{eff}}(\Phi, b_\mu) = \int d^3x\,\left(\mathrm{Tr}\,(D_\mu\Phi)^\dagger D^\mu\Phi + V(\Phi,\Phi^\dagger)\right)$
- $V(\Phi,\Phi^\dagger) = \frac{\lambda}{2}\mathrm{Tr}\,(\Phi^\dagger\Phi - a^2 I_N)^2 - \xi\big(\det\Phi + \det\Phi^\dagger\big) - \vartheta\mathrm{Tr}\,(\Phi^\dagger T_A\Phi T_A) + c$



A chain configuration, with n correlated instantons, linking a Wilson Loop $\mathcal{C}$

Effective model in 3D

Action

$$S_{\text{eff}}(\Phi, b_\mu) = \int d^3x \left(\text{Tr}\,(D_\mu\Phi)^\dagger D^\mu\Phi + V(\Phi,\Phi^\dagger)\right)$$
$$V(\Phi,\Phi^\dagger) = \tfrac{\lambda}{2}\text{Tr}\,(\Phi^\dagger\Phi - a^2 I_N)^2 - \xi\left(\det\Phi + \det\Phi^\dagger\right) - \vartheta\text{Tr}\,(\Phi^\dagger T_A \Phi T_A) + c$$

Polar Decomposition

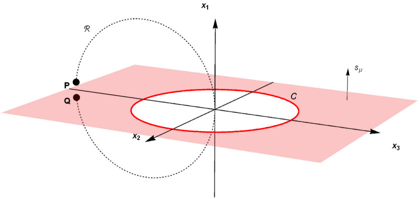
$$\Phi = PU$$
$$V(P,U) = \tfrac{\lambda}{2}\text{Tr}\,\left((P^2 - a^2 I_N)^2\right) - \xi\det P\,(\det U + \det U^\dagger) - \vartheta\text{Tr}\,\left(PT_A P U T_A U^\dagger\right)$$

Vacuum

$$P = vI_N\ ,\ U \in \mathcal{Z}_N = \left\{ e^{i\frac{2\pi n}{N}} I_N \,\middle|\, n = 0,1,2,...,N-1 \right\}$$
$$2\lambda N(v^2 - a^2) - 2\xi N v^{N-2} - \vartheta\left(N^2 - 1\right) = 0$$

Domain Walls

- Φ must be in the vacuum at **P** and **Q**
- $\lim_{x_1 \to -\infty} \Phi(x_1, x_2, x_3) = v\, I_N$
- $\lim_{x_1 \to +\infty} \Phi(x_1, x_2, x_3) = v\, e^{i2\pi \beta_e \cdot T}$
- $S_{\text{eff}} \approx \varepsilon A$
- $\varepsilon = \int dx \left(\mathrm{Tr}\,(\partial_x \Phi)^\dagger \partial_x \Phi + V(\Phi, \Phi^\dagger) \right)$



A ring $\mathcal{R}$ that goes through the center of the Wilson loop. The red surface is the one where s_μ is concentrated.

Soliton Ansatz

Profiles definitions

$$\Phi = (\eta I_N + \eta_0 \beta \cdot T)\, e^{i\theta\beta\cdot T} e^{i\alpha}$$

Profiles masses

$$\partial_x^2\, \delta f = M_f^2\, \delta f$$
$$M_\eta^2 = \lambda(3v^2 - a^2) - \xi(N-1)v^{N-2} - \vartheta\frac{N^2-1}{2N^2}$$
$$M_{\eta 0}^2 = \lambda(3v^2 - a^2) + \xi v^{N-2} + \frac{\vartheta}{2N^2}$$
$$M_\alpha^2 = N\xi v^{N-2}$$
$$M_\theta^2 = \frac{\vartheta}{2}$$

Casimir Law

$$\lambda a^2,\ \xi v^{N-2} >> \vartheta$$
$$\partial_x^2\, \theta = \frac{\vartheta}{2}\sin\theta$$
$$\varepsilon_k = 2v^2 \int \mathrm{Tr}\left(\partial_x S^\dagger \partial_x S\right) dx = v^2\,\frac{\beta_e\cdot\beta_e}{N} \int (\partial_x\theta)^2 dx$$

Ensemble in 4D

- Ensemble of worldsurfaces is still lacking. The resulting order parameter would be a string field $V(C)$
- $S_V = -\sum_C \sum_{p\in\eta(C)} \left[\bar{V}(C+p)U_p V(C) + \bar{V}(C-p)\bar{U}_p V(C)\right] + \sum_C m^2 \bar{V}(C)V(C)$
- $U_p = e^{ia^2 B_{\mu\nu}(p)}$
- In $m^2 < 0$, $V(C) \approx \omega \prod_{l\in C} V_l \ , \ V_l \in U(1)$.
- $B_{\mu\nu} \to \frac{2\pi k}{N} s_{\mu\nu}$
- In analogy with the 3D case, the resulting action is $S = \int d^4x \langle F_{\mu\nu}(\Lambda) - s_{\mu\nu}\rangle^2$
- Monopoles are represented by adjoint ψ_l

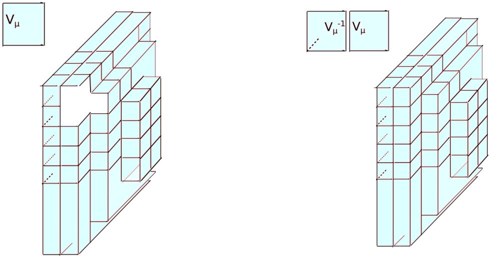
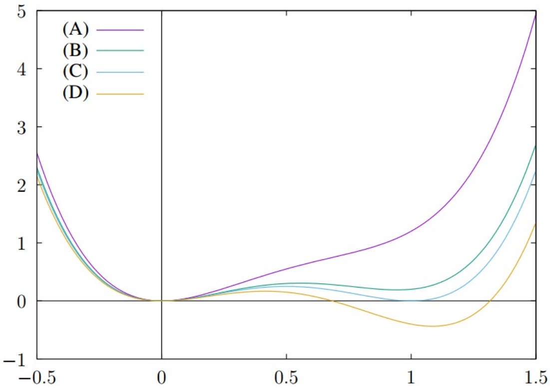


Figure: Configurations with open surfaces do not contribute while configurations of closed ones do.

Effective model in 4D

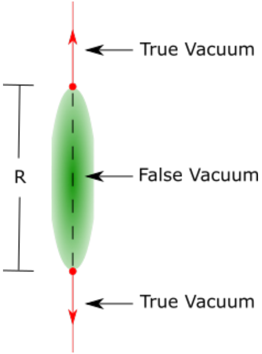
- $S = \int d^4x \left(\frac{1}{4}\langle F_{\mu\nu} - s_{\mu\nu}\rangle^2 + \frac{1}{2}\langle D_\mu\psi_I, D_\mu\psi_I\rangle + V_{\rm H}(\psi)\right)$
- $V_{\rm H}(\psi) = c + \frac{\mu^2}{2}\langle\psi_A,\psi_A\rangle + \frac{\kappa}{3}f_{ABC}\langle\psi_A\wedge\psi_B,\psi_C\rangle + \frac{\lambda}{4}\langle\psi_A\wedge\psi_B,\psi_A\wedge\psi_B\rangle$, $(N^2-1)^2$ perfis de Higgs
- $\Lambda_\mu = \frac{i}{g}S\partial_\mu S^{-1}$,
- $\psi_A = vST_AS^{-1}$, $v = -\frac{\kappa}{2\lambda} + \sqrt{\left(\frac{\kappa}{2\lambda}\right)^2 - \frac{\mu^2}{\lambda}}$
- Model is unstable for $\mu^2 < 0$



Potencial as a function of v in the regimes: (A) $\mu^2 > \frac{1}{4}\frac{\kappa^2}{\lambda}$, (B) $\frac{1}{4}\frac{\kappa^2}{\lambda} > \mu^2 > \frac{2}{9}\frac{\kappa^2}{\lambda}$, (C) $\mu^2 = \frac{2}{9}\frac{\kappa^2}{\lambda}$, (D) $\mu^2 < \frac{2}{9}\frac{\kappa^2}{\lambda}$

Flux tube ansatz

- $\Lambda_i = S\mathcal{A}_i S^{-1} + \frac{i}{g} S\partial_i S^{-1}$
- $\psi_A = h_{AB} S T_A S^{-1}$, $S = e^{i\varphi\beta\cdot T}$, $\beta = 2N\Lambda^k$
- $\mathcal{A}_i = \frac{a-1}{g}\partial_i\varphi\,\beta\cdot T \implies \Lambda_i = \frac{a}{g}\partial_i\varphi\,\beta\cdot T$
- $a(\rho\to\infty) = 1$, $a(\rho\to 0) = 0$
- $h_{AB}(\rho\to\infty) = v\delta_{AB}$
- $ST_qS^{-1} = T_q$, $ST_\alpha S^{-1} =$ $\cos(\alpha\cdot\beta\,\varphi)\,T_\alpha - \sin(\alpha\cdot\beta\,\varphi)\,T_{\overline{\alpha}}$
- $h_\alpha(\rho\to 0) = 0 \quad \text{if} \quad \alpha\cdot\beta \neq 0$



Flux tube between static charges (Wilson loop)

Case $k = 1$

Collective behavior

$$h_\alpha = h_{\overline{\alpha}} = \begin{cases} h_0, \text{ if } \alpha\cdot\beta = 0 \\ h, \text{ if } \alpha\cdot\beta = 1 \end{cases}, \quad h_{qp} = h_1 M_1|_{qp} + h_2 M_2|_{qp}\,, \quad h_2 = h_0$$

Form equations

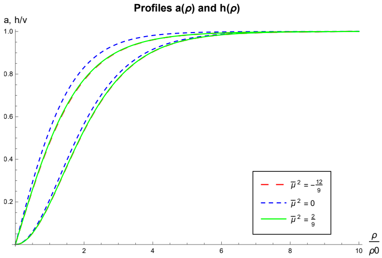
$$\frac{1}{\rho}\frac{\partial a}{\partial\rho} - \frac{\partial^2 a}{\partial\rho^2} = g^2 h^2 (1-a)$$

$$\nabla^2 h_1 = \mu^2 h_1 + (\kappa + \lambda h_1) h^2$$

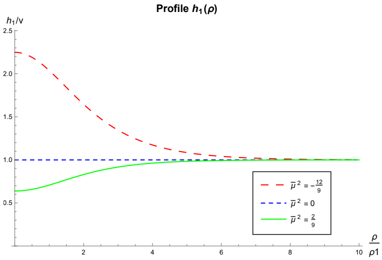
$$\nabla^2 h_0 = \mu^2 h_0 + \frac{h^2 + (N-1) h_0^2}{N} (\kappa + \lambda h_0)$$

$$\nabla^2 h = \mu^2 h + \frac{(1-a)^2}{\rho^2} h + \frac{\lambda}{2} h^3 + \frac{(N-2)}{2(N-1)} h h_0 (2\kappa + \lambda h_0) + \frac{(2\kappa + \lambda h_1)}{2(N-1)} h h_1$$

Numerical Solutions



Profiles $a(\rho)$ and $h(\rho)$ for different values of μ^2. The profile a is the one that goes linearly to 0 when $\rho \to 0$.



Profiles $h_1(\rho)$ for different values of μ^2.

Case $k > 1$

New Profiles

$$h_\alpha = \begin{cases} \tilde{h}_0, \text{ se } \alpha = \tilde{\alpha}_0 \\ h_0, \text{ se } \alpha = \alpha_0 \\ h, \text{ se } \alpha \cdot \beta = 1 \end{cases}$$

$$h_{qp} = h_1 M_1|_{qp} + h_2 M_2|_{qp} + h_3 M_3|_{qp}$$

$$h_2 = h_0 \ , \ h_3 = \tilde{h}_0$$

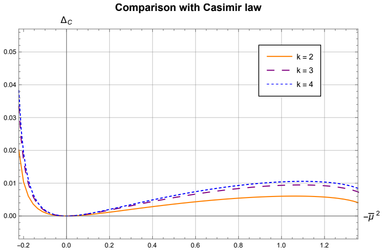
Energy

$$E = \int d^3x \, \frac{k(N-k)}{\rho^2} \left(\frac{|\nabla a|^2}{g^2} + h^2(1-a)^2 \right) + k(N-k) \left(|\nabla h|^2 + \mu^2 h^2 \right)$$
$$+ \frac{1}{2} \left(|\nabla h_1|^2 + \mu^2 h_1^2 \right) + \frac{(N-k)^2 - 1}{2} \left(|\nabla h_0|^2 + \mu^2 h_0^2 \right) + \frac{k^2 - 1}{2} \left(|\nabla \tilde{h}_0|^2 + \mu^2 \tilde{h}_0^2 \right)$$
$$+ \lambda \frac{k(N-k)}{4} h^4 + C_1(h_1, h_0, \tilde{h}_0) h^2 + C_2(h_1, h_0, \tilde{h}_0)$$

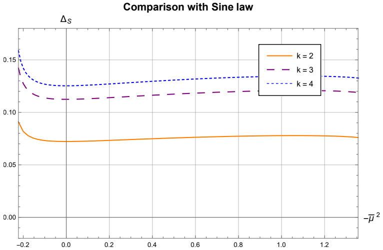
Casimir Law

At $\mu^2 = 0$, $E_k = k \frac{N-k}{N-1} E_{k=1}$

Numerical solutions



Plot of $\Delta_C(k)$, i.e. the relative deviation of the Casimir Law.



Plot of $\Delta_S(k)$, i.e. the relative deviation of the Sine Law. Notice the deviation is much bigger in the explored parameter region.

k-symmetric representations

Ansatz

$$\Lambda_i = S\mathcal{A}_i S^{-1} + \frac{i}{g} S\partial_i S^{-1} \ , \ \psi_A = h_{AB} S T_A S^{-1} \ , \ S = e^{i\varphi\beta\cdot T} \ , \ \beta = 2Nk\omega_1$$

Energy at $\mu^2 = 0$

$$\frac{E}{N-1} = \int d^3x \, \frac{k^2}{\rho^2}\left(\frac{|\nabla a|^2}{g^2} + h^2(1-a)^2\right) + \left(|\nabla h|^2 + \mu^2 h^2\right) + \frac{\lambda}{4}\left(h^2 - v^2\right)^2$$

In the BPS point $\lambda = g^2$, we have $E_k = kE_1$

BPS for a general irrep.

BPS equations

$$\zeta_\alpha = \frac{\psi_\alpha + i\psi_{\bar{\alpha}}}{\sqrt{2}}$$

$$B_1 = B_2 = 0 \ , \ D_3\psi_A = 0 \leftarrow \text{translational invariance}$$

$$D_+\zeta_\alpha = 0 \ , \ D_1\psi_q = D_2\psi_q = 0$$

$$B_3 = g \sum_{\alpha>0} \left(v\alpha|_q \psi_q - [\zeta_\alpha, \zeta_\alpha^\dagger] \right)$$

Ansatz

$$\Lambda_i = S\mathcal{A}_i S^{-1} + \frac{i}{g} S\partial_i S^{-1} \ , \ S = e^{i\varphi\beta\cdot T} \ , \ \beta = 2N\Lambda$$

$$\psi_q = T_q \ , \ \zeta_\alpha = SE_\alpha S^{-1}$$

$$\mathcal{A}_i = \sum_{l=1}^{N-1} \frac{a_l - d_l}{g} \partial_i\varphi \left(2N\Lambda^l\right) \cdot T$$

Asymptotic scaling law

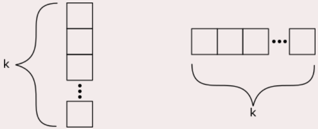
BPS Energy

$$\epsilon = \int d^2x \left(\frac{1}{2}\langle B_3, B_3\rangle + \sum_{\alpha>0} \langle D_i\zeta_\alpha^\dagger, D_i\zeta_\alpha\rangle + V_{\mathrm{H}}(\psi)\right)$$

$$\epsilon = 4N\pi g v^2 \lambda^D \cdot 2\delta \ , \ 2\delta = \sum_{j>i} \alpha_{ij}$$

Quadratic Casimir $\rightarrow C_2(\lambda^D) = \lambda^D \cdot \lambda^D + \lambda^D \cdot 2\delta$

Young Tableaux



Young tableaux for the k-A (left) and k-S (right) representations

$$\lambda^D = \sum_{l=1}^{N-1} d_l \lambda^{l\text{-A}} \ , \ d_i = m_i - m_{i+1}$$

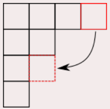
$N-$ality $\leftrightarrow$ # caixas mod N

$k-$Antisym.: $m_{i\leq 1} = 1$

$k-$Sym.: $m_1 = k$

Calculation of $\beta \cdot 2\delta$

Lowering boxes



Uma transformação que diminui $\beta \cdot 2\delta$.

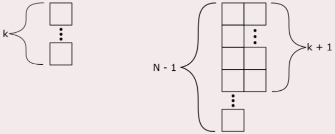
$$\Delta\beta \cdot 2\delta = N^2 \Delta n - \frac{2N}{N+1} \sum_{l=1}^{N-1} \Delta m_l \, l$$

For $\Delta n = 0$ and $I < J$

$$\Delta m_J = -\Delta m_I = 1$$

$$\Delta\beta \cdot 2\delta = \frac{2N}{N+1}(I - J) < 0$$

Upper-left justified tableaux



Antisymmetric tableaux with k boxes (left) and symmetric tableaux with $N + k$ boxes (right).

$$\Delta\beta \cdot 2\delta = \frac{2N}{N+1}(N - k) > 0$$

Generalized by induction

$k-$Antisym. has the least value of $\beta \cdot 2\delta$

Local gauge fixing

- Singer theorem
- Doesn't hold if the fixing is made by sectors and if those sectors are disjointed $\vartheta_\alpha \subset \{A_\mu\}$:
  $\{A_\mu\} = \cup_\alpha \vartheta_\alpha \ , \ \vartheta_\alpha \cap \vartheta_\beta = \emptyset \ \text{ se } \ \alpha \neq \beta$

The procedure

$$S_{\rm aux}[A, \psi]$$

$$\frac{\delta S_{\rm aux}}{\delta \psi_I} = 0 \text{ e } D_\mu \psi \to 0 \ , \ |x| \to \infty \Rightarrow \psi_I[A]$$

Polar decomposition: $\psi_I = S q_I S^{-1}$

Gauge transformation: $A_\mu \to A_\mu^U \ , \ q[A^U] = q[A] \ , \ S[A^U] = US[A]$

Fixing $S = S_0$ also fixes the gauge

Local gauge fixing

Example

$$\text{Ex.: } V_{\mathrm{H}}(\psi) = c + \frac{\mu^2}{2}\langle \psi_A, \psi_A \rangle + \frac{\kappa}{3} f_{ABC} \langle \psi_A \wedge \psi_B, \psi_C \rangle + \frac{\lambda}{4} \langle \psi_A \wedge \psi_B, \psi_A \wedge \psi_B \rangle$$

For $A_\mu = \frac{a}{g}\partial_i \varphi \beta \cdot T$, we know tat $\psi_A = h_{AB} S_0 T_B S_0^{-1}$, $S_0 = e^{i\varphi\beta \cdot T}$

Boundary conditions: $h_{AB}(\rho \to \infty) \to v\delta_{AB}$

Pure Module: $q_A = h_{AB} T_B \Rightarrow \sum_A [q_A, T_A] = i f_{ABC} h_{AB} T_C = 0$ since h_{AB} is symmetric by $A \leftrightarrow B$

Important requirements

- Uniqueness of the solution ψ_A
- Injectivity: $\psi(A^U) = \psi(A) \Rightarrow U \in Z(N)$
- Unique polar decomposition

Thanks for your attention!