CHAPTER 4

COUNTING STATISTICS AND ERROR PREDICTION

Radioactive decay is a random process. Consequently, any measurement which is based on observing the radiation emitted in nuclear decay is subject to some degree of statistical fluctuation. These inherent fluctuations represent an unavoidable source of uncertainty in all nuclear measurements, and often can be the predominant source of imprecision or error. The term *counting statistics* includes the framework of statistical analysis required to process the results of nuclear counting experiments and to make predictions about the expected precision of quantities derived from these measurements.

The value of counting statistics falls into two general categories. The first is to serve as a check on the normal functioning of a piece of nuclear counting equipment. Here a set of measurements is recorded under conditions in which all aspects of the experiment are held as constant as possible. Because of the influence of statistical fluctuations, these measurements will not all be the same but will show some degree of internal variation. The amount of this fluctuation can be quantified and compared with predictions of statistical models. If the amount of observed fluctuation is not consistent with predictions, one can conclude that some abnormality exists in the counting system. The second application is generally more valuable and deals with the situation in which we have only one measurement. We can then use counting statistics to predict its inherent statistical uncertainty and thus estimate a precision that should be associated with that single measurement.

The distinctions made in the organization of this chapter are a critical part of the topic. The confusion that often arises when the student is first introduced to counting statistics arises more from a failure to keep separate the concepts presented in Sections I and II below than from any other single cause. In Section I we will be careful to limit the discussion to methods used in the

characterization or organization of experimental data. We are not particularly concerned where these data come from, but rather are interested only in presenting the formal methods by which we can describe the amount of fluctuation displayed by the data. In Section II, we will discuss the separate topic of probabilistic mathematical models which can sometimes represent real measurement systems. For purposes of the discussion in Section II, however, we are concerned only with the structure and predictions of these models as mathematical entities. We will reserve, until Section III, the demonstration of how the statistical models can be matched to experimental data, resulting in the two common applications of counting statistics mentioned above. Finally, in Section IV, we will examine how the predicted statistical uncertainties contribute to the overall uncertainty in a numerical result which is calculated from nuclear counting data.

I. CHARACTERIZATION OF DATA

We will begin by assuming that we have a collection of N independent measurements of the same physical quantity:

$$x_1, x_2, x_3, \ldots x_i \ldots x_N$$

We will further assume that a single typical value x_i from this set can only assume integer values so that the data might represent, for example, a number of successive readings from a radiation counter for repeated time intervals of equal length. Two elementary properties of this data set are

"Sum": $$\Sigma \equiv \sum_{i=1}^{N} x_i \tag{4-1}$$

"Experimental mean": $$\bar{x}_e \equiv \Sigma / N \tag{4-2}$$

The experimental mean is written with the subscript to distinguish it from the mean of a particular statistical model which will be introduced later.

It is often convenient to represent the data set by a corresponding *frequency distribution function* $F(x)$. The value of $F(x)$ is the relative frequency with which the number appears in the collection of data. By definition

$$F(x) \equiv \frac{\text{number of occurrences of the value "}x\text{"}}{\text{number of measurements } (=N)} \tag{4-3}$$

The distribution is automatically normalized, that is,

$$\sum_{x=0}^{\infty} F(x) = 1 \tag{4-4}$$

TABLE 4-1. Example of Data Distribution Function

Data		Frequency Distribution Function	
8	14	$F(3)=1/20$	$=0.05$
5	8	$F(4)$	$=0$
12	8	$F(5)$	$=0.05$
10	3	$F(6)$	$=0.10$
13	9	$F(7)$	$=0.10$
7	12	$F(8)$	$=0.20$
9	6	$F(9)$	$=0.10$
10	10	$F(10)$	$=0.15$
6	8	$F(11)$	$=0.05$
11	7	$F(12)$	$=0.10$
		$F(13)$	$=0.05$
		$F(14)$	$=0.05$
		$\sum_{x=0}^{\infty} F(x)$	$=1.00$

As long as we do not care about the specific sequence of the numbers, the complete data distribution function $F(x)$ represents all the information contained in the original data set.

For purposes of illustration, Table 4-1 gives a hypothetical set of data consisting of 20 entries. Because these entries range from a minimum of 3 to a maximum of 14, the data distribution function will have nonzero values only between these extreme values of the argument x. The corresponding values of $F(x)$ are also shown in Table 4-1.

A plot of the data distribution function for the example is given in Fig. 4-1. Also shown directly above the plot is a horizontal bar graph of the original 20 numbers from which the distribution was derived. These data show an experimental mean of 8.8, and the distribution function is in some sense centered about that value. Furthermore, the relative shape of the distribution function indicates qualitatively the amount of internal fluctuation in the data set. For example, Fig. 4-2 shows the shape of the distribution functions corresponding to two extreme sets of data: one with large amounts of scatter about the mean and one with little. An obvious conclusion is that the width of the distribution function is a relative measure of the amount of fluctuation or scattering about the mean inherent in a given set of data.

It is possible to calculate the experimental mean by using the data distribution function, because the mean of any distribution is simply its first moment

$$\bar{x}_e = \sum_{x=0}^{\infty} x \cdot F(x) \tag{4-5}$$

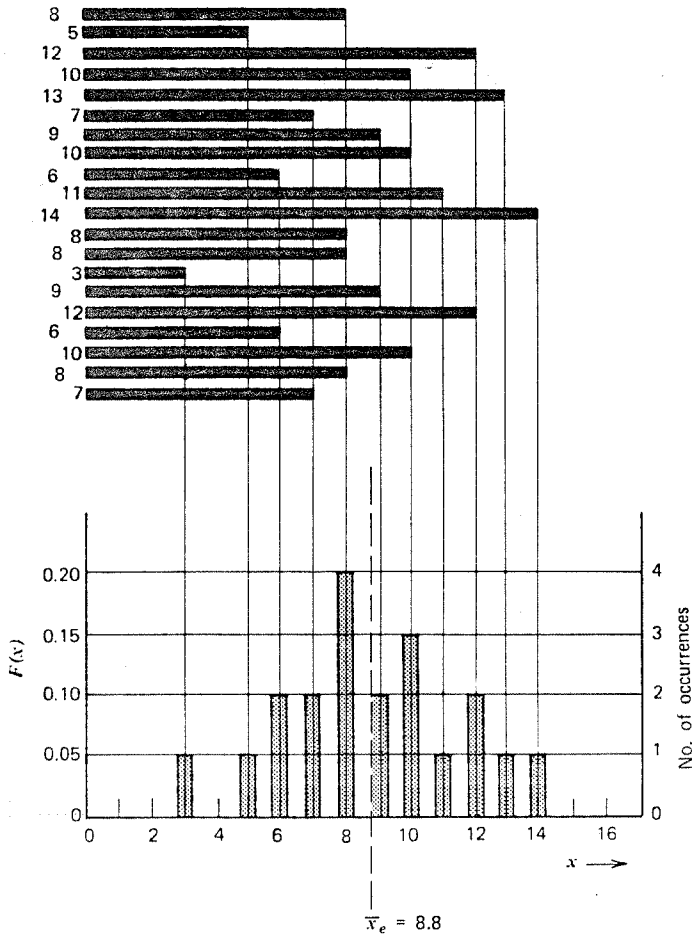


FIGURE 4-1. Distribution function for the data given in Table 4-1.

It is also possible to derive another parameter, known as the *sample variance*, which will serve to quantify the amount of internal fluctuation in the data set. The first step is to define the *deviation* of any data point as the amount by which it differs from the mean value

$$\epsilon_i \equiv x_i - \bar{x}_e \tag{4-6}$$

To illustrate, the example of the 20 numbers given in Table 4-1 is shown as the bar graph of Fig. 4-3*a*. The deviation of each of these values from the mean has been separately plotted on part *b* of the figure. There must be an equal

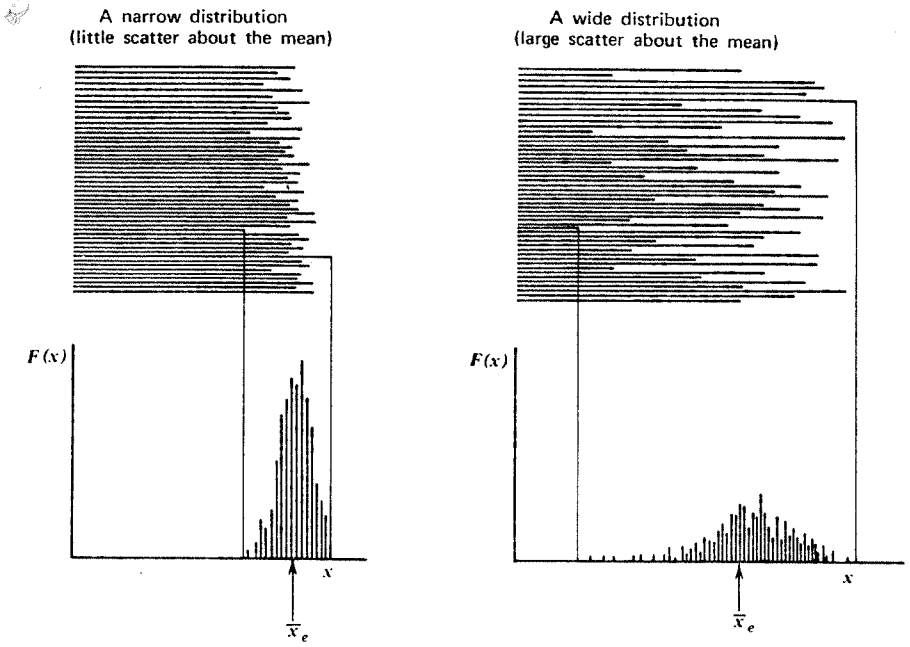


FIGURE 4-2. Distribution functions for two sets of data with differing amounts of internal fluctuation.

contribution of positive and negative deviations, so that

$$\sum_1^N \epsilon_i = 0 \tag{4-7}$$

If we take the square of each deviation, however, a positive number will always result. These are plotted for the example in Fig. 4-3*c*. We can now introduce the sample variance s^2 as

$$s^2 = \frac{1}{N-1} \sum_1^N \epsilon_i^2 \tag{4-8}$$

which will now serve as a single index of the degree of fluctuation inherent in the original data. As long as the number of data entries N is reasonably large, the sample variance is essentially the average value of the squared deviation of each data point. To be precise, the sample variance is more fundamentally defined as the average value of the deviation of each data point from the *true mean value* $\bar{x}$ which would be derived if an infinite number of data points were

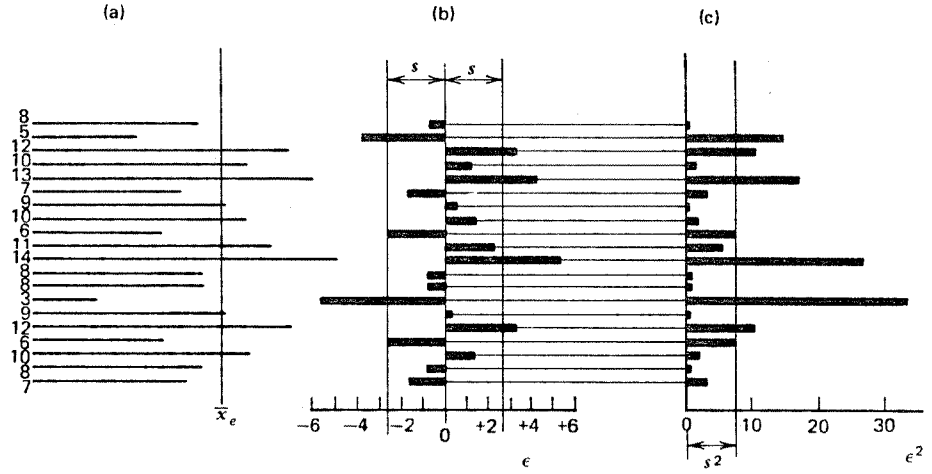


FIGURE 4-3. Part (a) shows a plot of the data given in Table 4-1. Corresponding values for the deviation ϵ and for ϵ^2 are shown in parts (b) and (c).

accumulated

$$s^2 \equiv \frac{1}{N} \sum_{i=1}^N (x_i - \bar{x})^2 \tag{4-9}$$

Because we cannot know $\bar{x}$ from a finite set of measurements, we use instead the value $\bar{x}_e$ derived from the data set itself to calculate values for the deviations. Use of the experimental, rather than true theoretical mean value, will tend to reduce the average deviation and therefore result in a smaller than normal variance. In statistical parlance, the number of degrees of freedom of the system has been reduced by one, and the -1 which appears in the denominator of Eq. 4-8 accounts for this self-minimizing effect.

The sample variance s^2 for the example of 20 numbers is shown graphically on Fig. 4-3*c*. Because it is essentially a measure of the average value of the squared deviations of each point, s^2 is an effective measure of the amount of fluctuation in the original data. A data set with a narrow distribution will have a small typical deviation from the mean, and therefore the value for the sample variance will be small. On the other hand, data with a large amount of fluctuation will have a wide distribution and a large value for typical deviations, and the corresponding sample variance will also be large. It is important to note that the sample variance is an absolute measure of the amount of internal scatter in the data, and does not, to first approximation, depend on the number of values in the data set. For example, if the data shown in Fig. 4-3 were extended by simply collecting an additional 20 values by the same process, we would not

expect the sample variance calculated for the extended collection of 40 numbers to be substantially different from that shown in Fig. 4-3.

We can also calculate the sample variance directly from the data distribution function $F(x)$. Because Eq. 4-9 indicates that s^2 is simply the average value of $(x-\bar{x})^2$ we can write that same average as

$$s^2 = \sum_{x=0}^{\infty} (x-\bar{x})^2 \cdot F(x) \tag{4-10}$$

Equation 4-10 is not introduced so much for its usefulness in computation as for the parallel it provides to a similar expression, Eq. 4-17, which will be introduced in a later discussion of statistical models. An expansion of Eq. 4-10 will yield the well-known result

$$s^2 = \overline{x^2} - (\bar{x})^2 \tag{4-11}$$

We now end our discussion of the organization of experimental data with two important conclusions:

1. Any set of data can be completely described by its frequency distribution function $F(x)$.
2. Two properties of this frequency distribution function are of particular interest: the experimental mean and the sample variance.

The experimental mean is given by Eq. 4-5 and is the value about which the distribution is centered. The sample variance is given by Eq. 4-10 and is a measure of the width of the distribution, or the amount of internal fluctuation in the data.

II. STATISTICAL MODELS

Under certain conditions, we can predict the distribution function that will describe the results of many repetitions of a given measurement. We will define a measurement as counting the number of *successes* resulting from a given number of *trials*. Each trial will be assumed to be a *binary* process in that only two results are possible: The trial is either a success or it is not a success. For everything that follows, we will also assume that the probability of success is a constant for all trials.

To show how these conditions apply to real situations, Table 4-2 gives three separate examples. The third example indicates the basis for applying the theoretical framework that follows to the case of counting nuclear radiation events. In this case a trial consists of observing a given radioactive nucleus for a period of time t, the number of trials is equivalent to the number of nuclei in the sample under observation, and the measurement consists of counting those nuclei that undergo decay. We will identify the probability of success of any one trial as p. In the case of radioactive decay, that probability is equal to $(1-e^{-\lambda t})$, where λ is the decay constant of the radioactive sample.

TABLE 4-2. Examples of Binary Processes

Trial	Definition of Success	Probability of Success $\equiv p$
Tossing a coin	"heads"	1/2
Rolling a die	"a six"	1/6
Observing a given radioactive nucleus for a time t	The nucleus decays during the observation	$1-e^{-\lambda t}$

Three specific statistical models will be introduced:

1. *The Binomial Distribution.* This is the most general model and is widely applicable to all constant-p processes. It is, unfortunately, computationally cumbersome in radioactive decay where the number of nuclei is always very large, and is used only rarely in nuclear applications. One example in which the binomial distribution must be used is in the examination of data acquired by counting a very short-lived radioisotope with high counting efficiency. In this case the criteria for applications of the models which follow are not met.
2. *The Poisson Distribution.* This model is a direct mathematical simplification of the binomial distribution under conditions that the success probability p is small. In practical terms, that condition implies that we have chosen an observation time which is small compared with the half life of the source, or that the detection efficiency is small. Then if we single out any given radioactive nucleus, the probability that it results in a recorded count in the observation time will be a very small number and the Poisson distribution will apply.
3. *The Gaussian or Normal Distribution.* The third important distribution is the Gaussian, which is a further simplification if the average number of successes is relatively large (say greater than 20 or 30). That condition will apply for any situation in which we accumulate more than a few counts during the course of the measurement. This is most often the case so that the Gaussian model is widely applicable to many problems in counting statistics.

It should be emphasized that all the above models become identical for processes with a small individual success probability p but with a large enough number of trials so that the expected mean number of successes is large.

A. The Binomial Distribution

The binomial distribution is the most general of the statistical models we will discuss. If n is the number of trials for which each trial has a success probability p, then the predicted probability of counting exactly x successes can be shown[1]

to be

$$P(x)=\frac{n!}{(n-x)!x!}p^x(1-p)^{n-x} \tag{4-12}$$

$P(x)$ is the predicted probability distribution function as given by the binomial distribution, and is defined only for integer values of n and x.

We will show one example of an application of the binomial distribution. Imagine that we have an honest die so that the numbers 1 through 6 are all equally probable. Let us define a successful roll as one in which any of the numbers 3, 4, 5, or 6 appear. Because these are four of the six possible results, the individual probability of success p is equal to $\frac{4}{6}$ or 0.667. We will now roll the die a total of ten times and record the number of rolls that result in success as defined above. The binomial distribution will now allow us to calculate the probability that exactly x out of the ten trials will be successful, where x can vary between 0 and 10. Table 4-3 gives the values of the predicted probability distribution from Eq. 4-12 for the parameters $p=\frac{2}{3}$ and $n=10$. The results are also plotted in Fig. 4-4. We see that 7 is the most probable number of successes

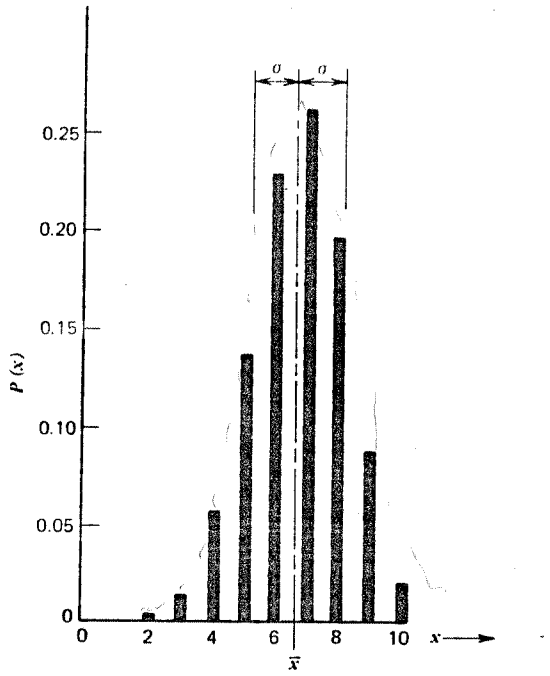


FIGURE 4-4. A plot of the binomial distribution for $p=2/3$ and $n=10$.

TABLE 4-3. Values of the Binomial Distribution for the Parameters $p=4/6$ or $2/3$, $n=10$

x	$P(x)$
0	0.00002
1	0.00034
2	0.00305
3	0.01626
4	0.05690
5	0.13656
6	0.22761
7	0.26012
8	0.19509
9	0.08671
10	0.01734
	$\sum_{x=0}^{10} P(x)=1.00000$

from the ten rolls of the die, with a probability of occurrence slightly greater than 1 out of 4. From the value of $P(0)$ we see that only twice out of 100,000 tests would we expect to see no successes from ten rolls of the die.

Some properties of the binomial distribution are important. First, the distribution is normalized

$$\sum_{x=0}^{n} P(x)=1 \tag{4-13}$$

Also we know that the average or mean value of the distribution is given by

$$\bar{x}=\sum_{x=0}^{n} x\cdot P(x) \tag{4-14}$$

If we now substitute Eq. 4-12 for $P(x)$ and carry out the summation, a remarkably simple result is derived

$$\bar{x}=p\cdot n \tag{4-15}$$

Thus we can calculate the expected average number of successes by multiplying the number of trials n by the probability p that any one trial will result in a success. In the example just discussed, we calculate an average number of successes as:

$$\bar{x}=pn=\left(\frac{2}{3}\right)(10)=6.67 \tag{4-16}$$

The mean value is obviously a very fundamental and important property of any predicted distribution.

It is also important to derive a single parameter which can describe the amount of fluctuation predicted by a given distribution. We have already defined such a parameter, called the sample variance, for a set of experimental data as defined in Eq. 4-10. By analogy we will now define a *predicted variance* σ^2 which will be a measure of the scatter about the mean predicted by a specific statistical model $P(x)$

$$\sigma^2 \equiv \sum_{x=0}^{n} (x-\bar{x})^2 \cdot P(x) \tag{4-17}$$

Conventionally, σ^2 is called the variance, and we will emphasize the fact that it is associated with a predicted probability distribution function by calling it a *predicted variance*. It is also conventional to define the *standard deviation* as the square root of σ^2. Recall that the variance is in some sense a typical value of the squared deviation from the mean. Therefore, σ represents a typical value for the deviation itself, hence the name "standard deviation."

Now if we carry out the summation indicated in Eq. 4-17 for the specific case of $P(x)$ given by the binomial distribution, the following result is obtained:

$$\sigma^2 = np(1-p) \tag{4-18}$$

Because $\bar{x} = np$, we can also write

$$\sigma^2 = \bar{x}(1-p) \tag{4-19}$$

$$\sigma = \sqrt{\bar{x}(1-p)} \tag{4-20}$$

We now have an expression which will give an immediate prediction of the amount of fluctuation inherent in a given binomial distribution in terms of the basic parameters of the distribution, n and p, where $\bar{x} = np$.

To return to the example of rolling a die given earlier, we defined success in such a way that $p = \frac{2}{3}$. We also assumed 10 rolls of the die for each measurement so that $n = 10$. For this example, the predicted mean number of successes is 6.67 and we can proceed to calculate the predicted variance

$$\sigma^2 = np(1-p) = (10)(0.667)(0.333) = 2.22 \tag{4-21}$$

By taking the square root we get the predicted standard deviation:

$$\sigma = \sqrt{\sigma^2} = \sqrt{2.22} = 1.49 \tag{4-22}$$

The significance of the standard deviation is illustrated in Fig. 4-4. The mean value of the distribution is shown as the dashed line, and one value of the

standard deviation is shown on either side of this mean. Because σ is a typical value for the difference between a given measurement and the true value of the mean, wide distributions will have large values for σ and narrow distributions will correspond to small values. The plot illustrates that the association of σ with the width of the distribution is not inconsistent with the example shown in Fig. 4-4.

B. The Poisson Distribution

Many categories of binary processes can be characterized by a low probability of success for each individual trial. Included are most nuclear counting experiments in which large numbers of nuclei (on the order of Avogadro's number) make up the sample or number of trials, whereas a relatively small fraction of these give rise to recorded counts. Similarly in a nuclear beam experiment, many nuclear particles from an accelerator might strike a target for every one recorded reaction product. Under these conditions the approximation that $p \ll 1$ will hold and some mathematical simplifications can be applied to the binomial distribution. It can be shown[2] that in this limit the binomial distribution reduces to the form

$$P(x) = \frac{(pn)^x e^{-pn}}{x!} \tag{4-23}$$

Because $pn = \bar{x}$ holds for this distribution as well as for the parent binomial distribution,

$$\boxed{P(x) = \frac{(\bar{x})^x e^{-\bar{x}}}{x!}} \tag{4-24}$$

which is now the familiar form of the Poisson distribution.

Recall that the binomial distribution requires values for two parameters: the number of trials n and the individual success probability p. We note from Eq. 4-24 that a significant simplification has occurred in deriving the Poisson distribution—only one parameter is required which is the product of n and p. This is a very useful simplification because now we need only know the mean value of the distribution in order to reconstruct its amplitude at all other values of the argument. That is a great help for processes in which we can in some way measure or estimate the mean value, but for which we haven't the slightest idea of either the individual probability or the size of the sample. Such is usually the case in nuclear measurements.

Some properties of the Poisson distribution follow directly. First, it is also a normalized distribution, or

$$\sum_{x=0}^{n} P(x) = 1 \tag{4-25}$$

We can also calculate the first moment or mean value of the distribution:

$$\bar{x} = \sum_{x=0}^{n} x \cdot P(x) = pn \tag{4-26}$$

which is the intuitively obvious result also obtained for the binomial distribution. The predicted variance of the distribution, however, differs from that of the binomial and can be evaluated from our prior definition

$$\sigma^2 \equiv \sum_{x=0}^{n} (x - \bar{x})^2 \cdot P(x) = pn \tag{4-27}$$

or noting the result from Eq. 4-26

$$\sigma^2 = \bar{x} \tag{4-28}$$

The predicted standard deviation is just the square root of the predicted variance, or

$$\sigma = \sqrt{\bar{x}} \tag{4-29}$$

Thus we see that the predicted standard deviation of any Poisson distribution is just the square root of the mean value which characterizes that same distribution. Note that the corresponding result obtained earlier for the binomial distribution (Eq. 4-20) reduces to the above result in the limit of $p \ll 1$ already incorporated into the Poisson assumptions.

We will again illustrate with an example. Suppose we randomly select a group of 1000 people and define our measurement as counting the number of current birthdays found among all members of that group. The measurement then consists of 1000 trials, each of which is a success only if a particular individual has his or her birthday today. If we assume a random distribution of birthdays, then the probability of success p is equal to $1/365$. Because p is much less than one in this example, we can immediately turn to the Poisson distribution to evaluate the probability distribution function which will describe the expected results from many such samplings of 1000 people. Thus, for our example,

$$p = \frac{1}{365} = 0.00274 \qquad \bar{x} = pn = 2.74$$

$$n = 1000 \qquad \sigma = \sqrt{\bar{x}} = 1.66$$

$$P(x) = \frac{\bar{x}^x e^{-\bar{x}}}{x!} = \frac{(2.74)^x e^{-2.74}}{x!}$$

x	$P(x)$
0	0.064
1	0.177
2	0.242
3	0.221
4	0.152
5	0.083
6	0.038
7	0.014
.	.
.	.
.	.

Recall that $P(x)$ gives the predicted probability that exactly x birthdays will be observed from a random sampling of 1000 people. The numerical values are plotted in Fig. 4-5 and show that $x = 2$ is the most probable result. The mean value of 2.74 is also shown in the figure, together with one value of the standard deviation of 1.66 on either side of the mean. The distribution is roughly centered about the mean value, although considerable asymmetry is evident for this low value of the mean. Again the size of the standard deviation gives some indication of the width of the distribution or the amount of scatter predicted by the distribution.

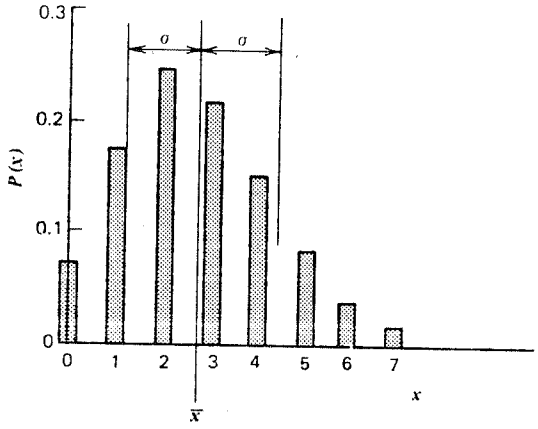


FIGURE 4-5. The Poisson distribution for a mean value $\bar{x} = 2.74$.

C. The Gaussian or Normal Distribution

The Poisson distribution holds as a mathematical simplification to the binomial distribution in the limit $p \ll 1$. If, in addition, the mean value of the distribution is large (say greater than 20) additional simplifications can generally be carried out[2] which lead to the *Gaussian* distribution:

$$P(x) = \frac{1}{\sqrt{2\pi\bar{x}}} \exp\left[-(x-\bar{x})^2/2\bar{x}\right] \tag{4-30}$$

This is again a pointwise distribution function defined only for integer values of x. It shares the following properties with the Poisson distribution:

1. It is normalized: $\sum_{x=0}^{\infty} P(x) = 1$
2. The distribution is characterized by a single parameter $\bar{x}$, which is given by the product np.
3. The predicted variance σ^2 as defined in Eq. 4-17 is again equal to the mean value $\bar{x}$.

We can again illustrate an example of a physical situation in which the Gaussian distribution is applicable. Suppose we return to the previous example of counting birthdays out of a group of randomly selected individuals, but now consider a much larger group of 10,000 people. For this example, $p = \frac{1}{365}$ and $n = 10{,}000$, so the predicted mean value of the distribution $\bar{x} = np = 27.4$. Because the predicted mean is larger than 20, we can turn to the Gaussian distribution for the predicted distribution of the results of many measurements, each of which consists of counting the number of birthdays found in a different group of 10,000 people. The predicted probability of observing a specific count x is then given by

$$P(x) = \frac{1}{\sqrt{2\pi(27.4)}} e^{-\frac{(x-27.4)^2}{54.8}} \tag{4-31}$$

and the predicted standard deviation for the example is:

$$\sigma = \sqrt{\bar{x}} = \sqrt{27.4} = 5.23 \tag{4-32}$$

These results are shown graphically in Fig. 4-6*a*.

Two important observations can be made at this point about the Gaussian distribution:

1. The distribution is symmetric about the mean value $\bar{x}$. Therefore $P(x)$ depends only on the absolute value of the deviation of any value x from the mean, defined as $\epsilon \equiv |x - \bar{x}|$.

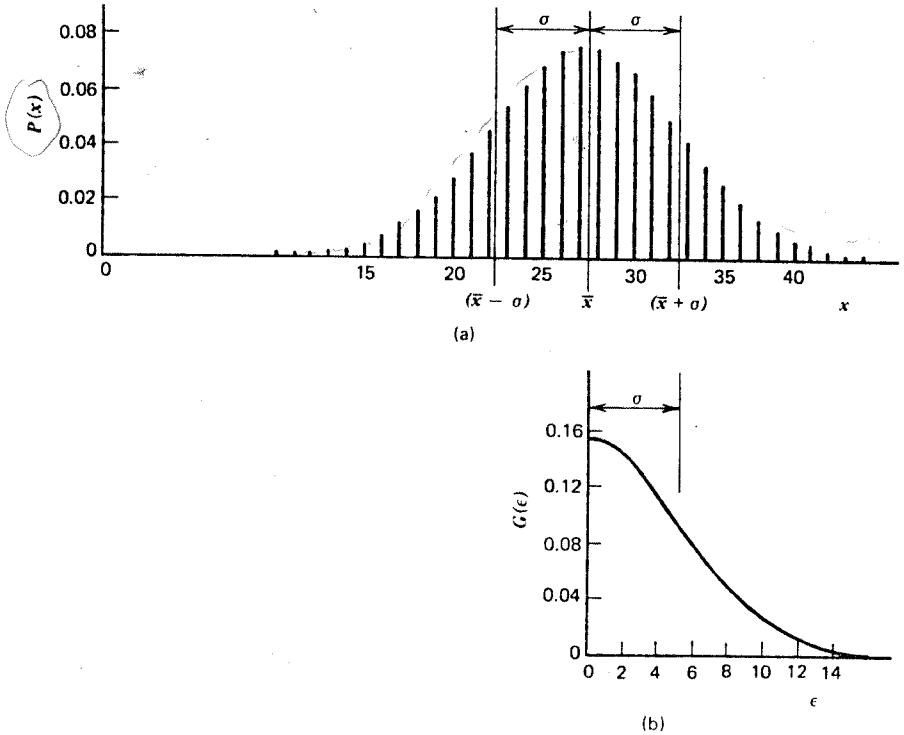


FIGURE 4-6. Part (a) shows the discrete Gaussian distribution for a mean value $\bar{x} = 27.4$. Part (b) is a plot of the corresponding continuous form of the Gaussian.

2. Because the mean value $\bar{x}$ is large, values of $P(x)$ for adjacent values of x are not greatly different from each other. In other words, the distribution is slowly varying.

These two observations suggest a recasting of the distribution as an explicit function of the deviation ϵ (rather than of x) and as a continuous function (rather than a pointwise discrete function). These changes are accomplished by rewriting the Gaussian distribution as

$$\boxed{G(\epsilon) = \sqrt{\frac{2}{\pi\bar{x}}}\, e^{-\epsilon^2/2\bar{x}}} \tag{4-33}$$

where $G(\epsilon)d\epsilon$ is now defined as the differential probability of observing a deviation in $d\epsilon$ about ϵ. Furthermore, because $\sigma = \sqrt{\bar{x}}$:

$$G(\epsilon) = \sqrt{\frac{2}{\pi\bar{x}}}\, e^{-\epsilon^2/2\sigma^2} \tag{4-34}$$

Comparing Eq. 4-33 with Eq. 4-30, we note a factor of two which has entered in $G(\epsilon)$ because there are two values of x for every value of the deviation ϵ.

Figure 4-6*b* shows the continuous form of the Gaussian distribution for the same example chosen to illustrate the discrete case. Comparing Fig. 4-6*a* and 4-6*b*, the scale factors for each abscissa are the same but the origin for Fig. 4-6*b* has been shifted to illustrate that a value of zero for the deviation ϵ corresponds to the position of the mean value $\bar{x}$ on Fig. 4-6*a*. If a factor two difference in the relative ordinate scales is included as shown, then the continuous distribution $G(\epsilon)$ represents the smooth curve which connects the pointwise values plotted in Fig. 4-6*a*.

Because we are now dealing with a continuous function, we must redefine some properties of the distribution as shown in Fig. 4-7. It should be particularly noted that quantities of physical interest now involve integrals of the distribution between set limits, or *areas* under the curve, rather than sums of discrete values.

Referring to Eq. 4-34, it is evident that the value of the exponential factor in the distribution depends only on the ratio of ϵ to σ. Therefore, all Gaussian curves (regardless of the value of $\bar{x}$ or σ) will have the same shape provided the scale factor for the deviation ϵ is chosen in units of the standard deviation σ. This universal curve is illustrated in Fig. 4-8.

From the definitions given in Fig. 4-7, the probability that a typical deviation ϵ predicted by a Gaussian distribution will be less than a specific value ϵ_0 is given by the integral $\int_0^{\epsilon_0} G(\epsilon)\,d\epsilon \equiv f(\epsilon_0)$. The value of this integral is illustrated by the shaded area in Fig. 4-8. Provided ϵ_0 is chosen in units of the standard deviation σ, $f(\epsilon_0)$ becomes independent of all other parameters of the distribution and will be a universal property of all Gaussian distributions. Tabular values for $f(\epsilon_0)$ can be found in most collections of statistical tables, and some selected entries are shown in Table 4-4. This function gives the probability that a random sample from a Gaussian distribution will show a deviation from the true mean value which is less than the assumed value of ϵ_0. For example, we can conclude that 68.3 percent of all samples will deviate from the true mean by less than one value of the standard deviation.

TABLE 4-4. Probability of Occurrence of Given Deviations Predicted by the Gaussian Distribution

ϵ_0	$f(\epsilon_0)$
0	0
$0.674\,\sigma$	0.500
σ	0.683
$1.64\,\sigma$	0.900
$1.96\,\sigma$	0.950
$2.58\,\sigma$	0.990
$3.00\,\sigma$	0.997

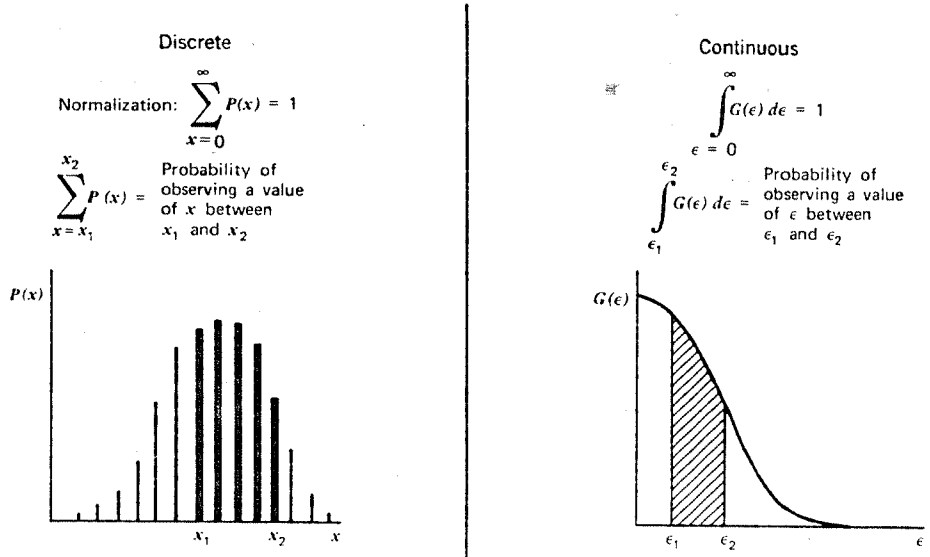


FIGURE 4-7. Comparison of the discrete and continuous forms of the Gaussian distribution.

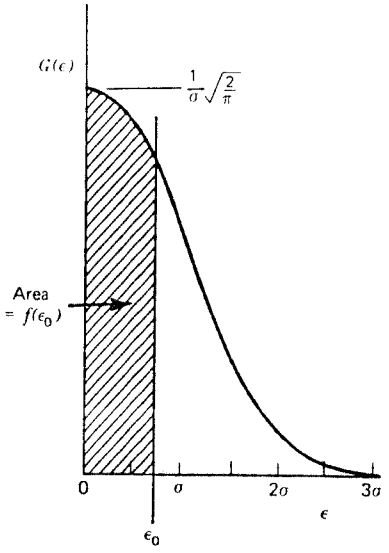


FIGURE 4-8. A plot of the general Gaussian curve.

III. APPLICATIONS OF STATISTICAL MODELS

The first two sections of this chapter have dealt with independent topics: the organization of experimental data in Section I, and the structure of certain statistical models in Section II. The practical uses of statistical analysis will now be illustrated by bringing together these two separate topics.

There are two major applications of counting statistics in nuclear measurements. The first of these we will call "Application A" and involves the use of statistical analysis to determine whether a set of multiple measurements of the same physical quantity shows an amount of internal fluctuation which is consistent with statistical predictions. The usual motivation here is to determine whether a particular counting system is functioning normally. Although this is a useful application, a far more valuable contribution of counting statistics arises in situations in which we have only a single experimental measurement. In "Application B" we will examine the methods available to make a prediction about the uncertainty one should associate with that single measurement to account for the unavoidable effects of statistical fluctuations.

Application A

Checkout of the counting system to see if observed fluctuations are consistent with expected statistical fluctuation.

A common quality control procedure in many counting laboratories is to periodically (perhaps once a month) record a series of 20 to 50 successive counts from the detector system while keeping all experimental conditions as constant as possible. By applying the analytical procedures to be described here, it can be determined whether the internal fluctuation shown by these multiple measurements is consistent with the amount of fluctuation expected if statistical fluctuations were the only origin. In this way abnormal amounts of fluctuation can be detected which could indicate malfunctioning of some portion of the counting system.

Figure 4-9 shows the chain of events that characterizes this application of counting statistics. Properties of the experimental data are confined to the left half of the figure, whereas on the right-hand side are listed properties of an appropriate statistical model. We start in the upper left corner with the collection of N independent measurements of the same physical quantity. These might be, for example, successive one-minute counts from a detector. Using the methods outlined in Section I, we can characterize the data in several ways. The data distribution function $F(x)$ as defined in Eq. 4-3 can be compiled. From this distribution, the mean value $\bar{x}_e$ and the sample variance s^2 can be computed by the formulas given in Eqs. 4-5 and 4-10. Recall that the mean value $\bar{x}_e$ gives the value about which the distribution is centered, whereas the sample variance s^2 is a quantitative measure of the amount of fluctuation present in the collection of data.

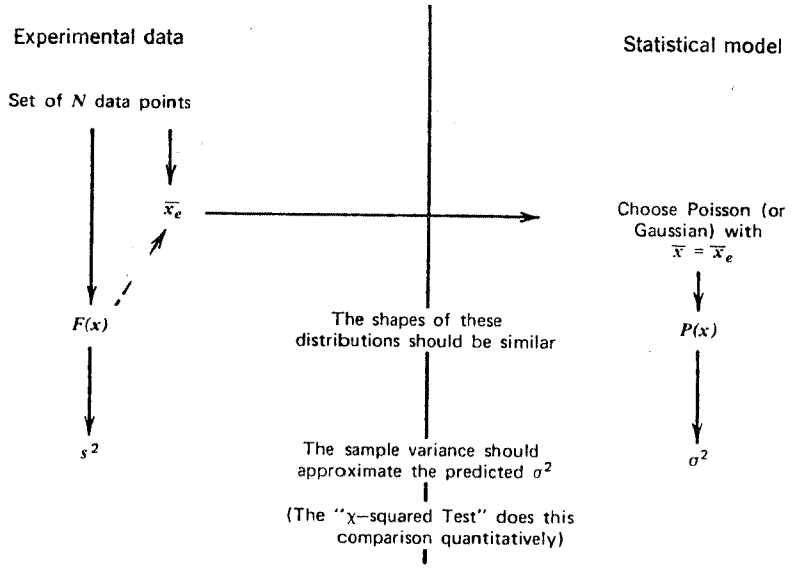


FIGURE 4-9. An illustration of "Application A" of counting statistics—the inspection of a set of data for consistency with a statistical model.

We now are faced with the task of matching these experimental data with an appropriate statistical model. Almost universally we will want to match to either a Poisson or Gaussian distribution (depending on how large the mean value is), either of which is fully specified by its own mean value $\bar{x}$. What should we choose for $\bar{x}$? We would be rather foolish if we chose any value other than $\bar{x}_e$, which is our only estimate of the mean value for the distribution from which the data have been drawn. Setting $\bar{x} = \bar{x}_e$ then provides the bridge from left to right in the figure, so that we now have a fully specified statistical model. If we let $P(x)$ represent the Poisson or Gaussian distribution with $\bar{x} = \bar{x}_e$, then the measured data distribution function $F(x)$ should be an approximation to $P(x)$ provided the statistical model accurately describes the distribution from which the data have arisen. One method of carrying out a comparison at this level is simply to make a superimposed plot of $F(x)$ and $P(x)$ and then to compare the shape and amplitude of the two distributions.

But such a comparison of two functions is, as yet, only qualitative. It is desirable to extract a single parameter from each distribution so that they can be compared quantitatively. The most fundamental parameter is the mean value, but these have already been matched and are the same by definition. A second parameter of each distribution is the variance, and we can carry out the desired quantitative comparison by noting the predicted variance σ^2 of the statistical model and comparing with the measured

sample variance s^2 of the collection of data. If the data are actually characterized by the statistical model and show a degree of internal fluctuation which is consistent with statistical prediction, these two variance values should be about the same. We will introduce the "Chi-squared test" as a systematic way of carrying out this comparison from which quantitative conclusions can be drawn.

To illustrate the direct comparison of the data distribution function with the predicted probability distribution function, we will return to the example of data given in Table 4-1. In Fig. 4-10 the data distribution function has been replotted as the solid vertical bars. The mean value for these data was calculated to be $\bar{x}_e = 8.8$, so the transition to the appropriate statistical model will be made by assuming its mean value to be $\bar{x} = 8.8$. Because the mean value is not large, we are prohibited from using the Gaussian distribution and we will therefore use the Poisson as the assumed statistical model. The points on Fig. 4-10 are the values of the predicted distribution function of the Poisson distribution for a mean value of 8.8. Because the Poisson is defined only for discrete values of x, the continuous curve is drawn only to connect the points for visual reference.

At this point a comparison of the two distributions is difficult. Because relatively little experimental data was gathered (20 measurements) the value of $F(x)$ at each point is subject to rather large fluctuations. One

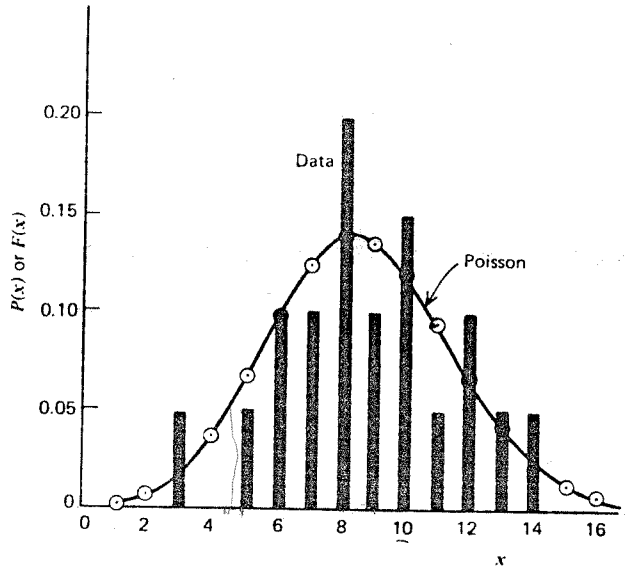


FIGURE 4-10. A direct comparison of experimental data (from Table 4-1) with predictions of a statistical model (the Poisson distribution for $\bar{x} = 8.8$).

would expect that, if more data were gathered, the fluctuations would diminish and the data distribution function $F(x)$ would adhere more and more closely to the predicted probability distribution function $P(x)$, provided the data are indeed a true sample from the predicted statistical model. From Fig. 4-10 we can only say that the experimental data are not grossly at variance with the prediction.

To take the comparison one step further, we would now like to intercompare the value of the sample variance and the predicted variance from the statistical model. The sample variance calculated from Eq. 4-8 for the same set of data is found to be

$$s^2 = 7.36$$

Because the assumed statistical model is the Poisson distribution, the predicted variance is given by

$$\sigma^2 = \bar{x} = 8.80$$

These two results show that there is less fluctuation in the data than would be predicted if the data were a perfect sample from a Poisson distribution of the same mean. With a limited sample size, however, one would not expect these two parameters to be precisely the same and a more quantitative test is required to determine whether the observed difference is really significant. This function is provided by the "Chi-squared test."

Chi-squared is simply another parameter of the experimental data distribution and is defined as

$$\chi^2 \equiv \frac{1}{\bar{x}_e} \sum_{i=1}^{N} (x_i - \bar{x}_e)^2 \tag{4-35}$$

where the summation is taken over each individual data point x_i. Chi-squared is closely related to the sample variance and the two are related by

$$\chi^2 = \frac{(N-1)s^2}{\bar{x}_e} \tag{4-36}$$

Now if the amount of fluctuation present in the data is closely modeled by the Poisson distribution, then $s^2 \cong \sigma^2$. But we know that for the Poisson distribution, $\sigma^2 = \bar{x}$. Furthermore, we have chosen $\bar{x}$ to be equal to $\bar{x}_e$. Therefore, the degree to which the ratio $s^2/\bar{x}$ deviates from unity is a direct measure of the extent to which the observed sample variance differs from the predicted variance. Now referring to Eq. 4-36, the degree to which χ^2 differs from $(N-1)$ is a corresponding measure of the departure of the data from predictions of the Poisson distribution. Chi-squared

TABLE 4-5. Portion of a Chi-Squared Distribution Table

Statistical Degrees of Freedom	No. of Measurements N	$p=0.8$	0.7	0.6	0.5
18	19	12.85	14.44	15.89	17.33
19	20	13.72	15.35	16.85	18.33
20	21	14.58	16.26	17.80	19.34

distribution tables may be found (e.g., Ref. 3) which are generally cast in the form shown in Table 4-5. The column on the left indicates the number of statistical degrees of freedom in the system. (This is one less then the number of independent measurements used to derive the value of χ^2 because $\bar{x}_e$ has been calculated from the same set of data.) Each column in the table is headed by a specific value of p, defined as the probability that a random sample from a true Poisson distribution would have a larger

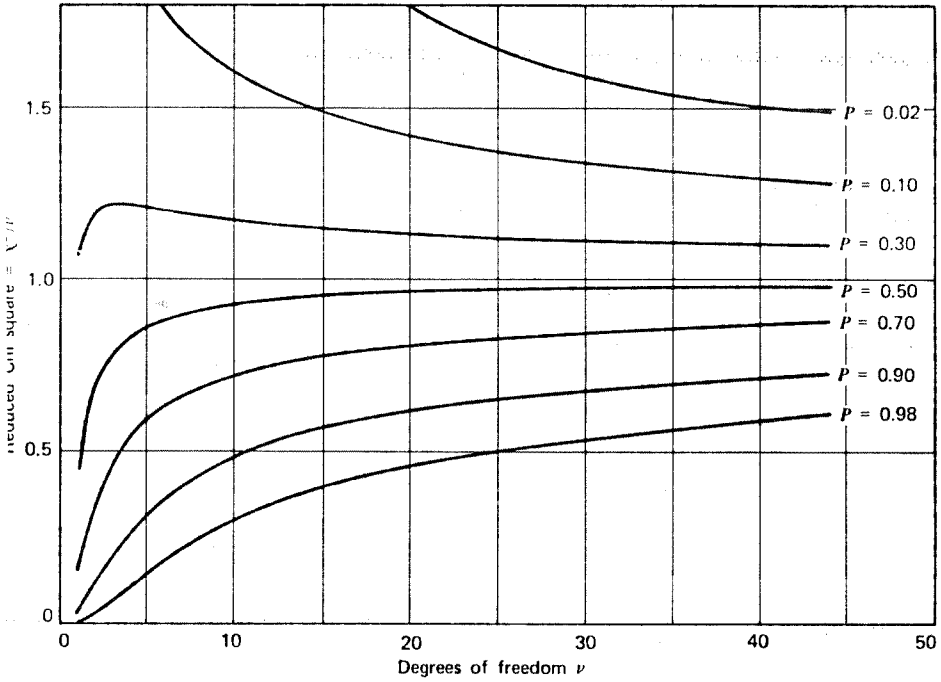


FIGURE 4-11. A plot of the Chi-square distribution. For each curve, p gives the probability that a random sample of N numbers from a true Poisson distribution would have a larger value of χ^2/ν than that of the ordinate. For data for which the experimental mean is used to calculate χ^2, the number of degrees of freedom $\nu = N-1$.

value of χ^2 than the specific value shown in the table. Very low probabilities (say less than 0.02) indicate abnormally large fluctuations in the data, whereas very high probabilities (greater than 0.98) indicate abnormally small fluctuations. A perfect fit to the Poisson would yield a probability of 0.50, whereas the somewhat arbitrary limits listed above indicate situations in which the counting system may be displaying either abnormally large fluctuations (which is the usual type of malfunction) or data that is too regular and shows abnormally small fluctuations. Figure 4-11 gives a plot of the χ^2 distribution for a wider range of the parameters involved.

For the illustrative example given above, we calculate a χ^2 value of 15.89. From Table 4-5 for $N=20$ we find (by interpolation) a value of $p=0.66$. Because that probability is neither very large nor very small, we would conclude that the equipment used to generate the set of numbers originally shown does not give rise to abnormal fluctuations.

Application B
Estimation of the precision of a single measurement.

A more valuable application of counting statistics applies to the case in which we have only a single measurement of a particular quantity, and wish to associate a given degree of uncertainty with that measurement. To state the objective in another way, we would like to have some estimate of the sample variance to be expected if we were to repeat the measurement many times. The square root of the sample variance should be a measure of the typical deviation of any one measurement from the true mean value, and thus will serve as a single index of the degree of precision one should associate with a typical measurement from that set. Because we have only a single measurement, however, the sample variance cannot be calculated directly but must be estimated by analogy with an appropriate statistical model.

The process is illustrated in Fig. 4-12. Again, the left half of the figure deals only with experimental data, whereas the right half deals only with the statistical model. We start in the upper-left corner with a single measurement, "x". If we make the assumption that the measurement has been drawn from a population whose theoretical distribution function is predicted by either a Poisson or Gaussian distribution, then we must match an appropriate theoretical distribution to the available data. For either model we must start with a value for the mean $\bar{x}$ of the distribution. Because the value of our single measurement "x" is the only information we have about the theoretical distribution from which it has been drawn, we have no real choice other than to assume that the mean of the distribution is equal to the single measurement, or $\bar{x}=x$. Having now obtained an assumed value for $\bar{x}$, the entire predicted probability distribu-

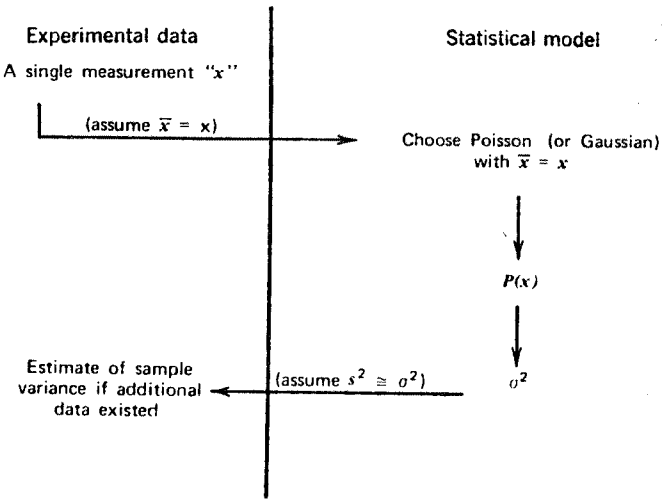


FIGURE 4-12. An illustration of "Application B" of counting statistics—prediction of the precision to be associated with a single measurement.

tion function $P(x)$ is defined for all values of x. We can also immediately find a value for the predicted variance σ^2 of that distribution. We can then use the association that, if the data are drawn from the same distribution, an estimate of the sample variance s^2 of a collection of such data should be given by σ^2. Through this process we have therefore obtained an estimate for the sample variance of a repeated set of measurements that do not exist, but which represent the expected results if the single measurement were to be repeated many times.

The conclusion we reach can then be stated as follows:

The expected sample variance	$s^2 \cong \sigma^2$	of the statistical model from which we think the measurement "x" is drawn
	$= \bar{x}$	provided the model is either Poisson or Gaussian
	$\cong x$	because "x" is our only measurement on which to base an estimate of $\bar{x}$

We therefore conclude that

$$\sqrt{s^2} \cong \sigma = \sqrt{x}$$

is our best estimate of the deviation from the true mean which should typify our single measurement "x".

This conclusion can be stated somewhat more quantitatively provided the assumed probability distribution function is a Gaussian (x is large). Then the range of values $x \pm \sigma$ or $x \pm \sqrt{x}$ will contain the true mean $\bar{x}$ with 68 percent probability. This conclusion follows directly from earlier statements about the shape of the Gaussian curve. It is conventional to quote the uncertainty or "error" of a single measurement as simply one value of the standard deviation σ. If we quote a larger uncertainty, then the probability of including the true mean within the quoted interval is increased, and vice versa.

To illustrate, assume we have a single measurement $x = 100$. Then

$$\sigma = \sqrt{x} = \sqrt{100} = 10$$

Because our best estimate of the mean value of the distribution from which this measurement was drawn (the measurement itself) is large, we can assume that the parent distribution is a Gaussian. From the shape of the Gaussian curve (see Table 4-4) we can then construct Table 4-6 for the specific example. The table gives various options available in quoting the uncertainty to be associated with our single measurement. The conventional choice is to quote the measurement plus or minus one value of the standard deviation σ, or 100 ± 10. This interval is expected to contain the true mean value $\bar{x}$ with a probability of 68 percent. If we wish to increase the probability that the true mean is included, we can do so only by expanding the interval or error associated with the measurement. For example, to achieve a 99 percent probability that the true mean is included, the interval must be expanded to 2.58 σ, or the range 100 ± 25.8 for our example. Unless otherwise stated, the errors quoted with a particular nuclear measurement normally represent one standard deviation.

TABLE 4-6. Examples of Error Intervals for a Single Measurement $x = 100$

Interval		Probability that the true mean $\bar{x}$ is included
$x \pm .67\sigma$	93.3–106.7	50%
$x \pm \sigma$	90 –110	68%
$x \pm 1.64\sigma$	83.6–116.4	90%
$x \pm 2.58\sigma$	74.2–125.8	99%

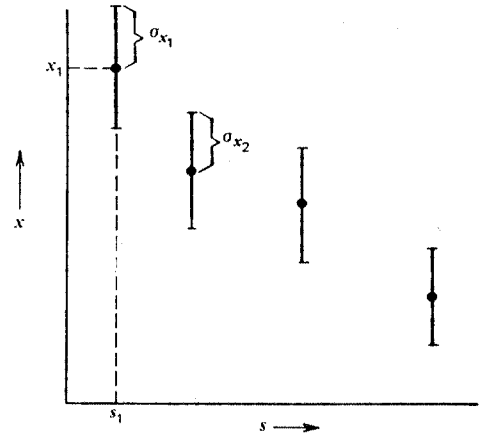


FIGURE 4-13. A graphical display of "error bars" associated with experimental data.

The *fractional standard deviation*, defined as σ/x, of a simple counting measurement is given by $\sqrt{x}/x$, or $1/\sqrt{x}$. Thus the total number of recorded counts x completely determines the fractional error to be associated with that measurement. If 100 counts are recorded, the fractional standard deviation is 10 percent, whereas it can be reduced to 1 percent only by increasing the total counts recorded to 10,000. For events occurring at a constant rate, this relationship implies that the time required to achieve a given fractional error will increase as the inverse square of the desired statistical precision.

When a set of measurements is presented graphically, the estimated errors associated with each measurement are often also displayed on the same graph. Figure 4-13 gives a hypothetical set of measurements of a quantity x as a function of some other variable or parameter s. The measured data are presented as points, whereas the error associated with each point is indicated by the length of the "error bar" drawn around each point. It is conventional to show the length of the error bar equal to one value of σ on either side of the point, or the total length of the error bar equal to 2σ. Under these conditions, if one were to attempt a fit of an assumed functional behavior $x=f(s)$, the fitted function should pass through 68 percent (or roughly two-thirds) of all the error bars associated with the data.

Caution

All the conclusions we have drawn apply *only* to a measurement of a number of successes (number of heads in coin tosssing, number of birthdays, etc.). In radioactive decay or nuclear counting, we may directly apply $\sigma=\sqrt{x}$ *only* if x represents a counted number of successes, that is, a number events over a given observation time recorded from a detector.

The vast majority of mistakes made in the use of counting statistics results from the misapplication of the above relationship.

One *cannot* associate the standard deviation σ with the square root of any quantity which is not a directly measured number of counts. For example, the association does *not* apply to:

1. counting *rates*
2. *sums* or *differences* of counts
3. *averages* of independent counts
4. any *derived* quantity.

In all these cases the quantity is *calculated* as a function of the number of counts recorded in a given experiment. The error to be associated with that quantity must then be calculated according to the methods outlined in the next section.

IV. ERROR PROPAGATION

In a typical nuclear measurement, one is seldom interested in the unprocessed data consisting of the number of counts over a particular interval. More often the data are processed through multiplication, addition, or other functional manipulation to arrive at a derived number of more direct interest. We must then be concerned with the manner in which the error associated with the original number of counts propagates through these calculations and is reflected as a corresponding uncertainty in the derived quantity. It can be shown[3] that if the errors are individually small and symmetric about zero, a general result can be obtained for the expected error to be associated with any quantity which is calculated as a function of any number of independent variables. If $x,y,z,\ldots$ are directly measured counts or related variables for which we know $\sigma_x,\sigma_y,\sigma_z,\ldots$, then the standard deviation for any quantity u derived from these counts can be calculated from

$$\sigma_u^2=\left(\frac{\partial u}{\partial x}\right)^2\sigma_x^2+\left(\frac{\partial u}{\partial y}\right)^2\sigma_y^2+\left(\frac{\partial u}{\partial z}\right)^2\sigma_z^2+\cdots \tag{4-37}$$

where $u=u(x,y,z,\ldots)$ represents the derived quantity. Equation 4-37 is generally known as the *error propagation formula* and is applicable to almost all situations in nuclear measurements. The variables $x,y,z,\ldots$, however, must be chosen so that they are truly independent in order to avoid the effects of correlation. The same specific count should not contribute to the value of more than one such variable. The use of Eq. 4-37 can be illustrated by application to some simple cases.

Case 1. Sums or Differences of Counts

If we define

$$u = x + y \quad \text{or} \quad u = x - y$$

then

$$\frac{\partial u}{\partial x} = 1 \quad \text{and} \quad \frac{\partial u}{\partial y} = \pm 1$$

Application of Eq. 4-37 yields

$$\sigma_u^2 = (1)^2\sigma_x^2 + (\pm 1)^2\sigma_y^2$$

or

$$\boxed{\sigma_u = \sqrt{\sigma_x^2 + \sigma_y^2}} \tag{4-38}$$

A common application of this case arises when counts due to a radioactive source must be corrected by subtracting an appropriate background count. If we assume equal counting times, then

$$\text{net counts} = \text{total counts} - \text{background counts}$$

or

$$u = x - y$$

Because both x and y are directly measured numbers of counts (or successes), the expected standard deviation of each is known to be its own square root. The object is to deduce the expected standard deviation of the net counts, a derived number. Because a simple difference is involved, the answer will be given by Eq. 4-38.

To illustrate by example, suppose we have recorded the following data for equal counting times

	total counts	$= x = 1071$
	background counts	$= y = \underline{\ 521}$
then		
	net counts	$= u = \ 550$

We know *a priori*

$$\sigma_x = \sqrt{x} = \sqrt{1071}$$
$$\sigma_y = \sqrt{y} = \sqrt{521}$$

Thus

$$\sigma_u = \sqrt{\sigma_x^2 + \sigma_y^2} = \sqrt{x + y} = \sqrt{1592} = 39.9$$

We would then quote the result plus or minus one standard deviation as

$$\text{net counts} = 550 \pm 39.9$$

Case 2. Multiplication or Division by a Constant

If we define

$$u = Ax$$

where A is a constant (no associated uncertainty), then

$$\frac{\partial u}{\partial x} = A$$

and application of Eq. 4-37 gives

$$\boxed{\sigma_u = A\sigma_x} \tag{4-39}$$

Similarly, if

$$v = \frac{x}{B}$$

where B is also a constant, then

$$\boxed{\sigma_v = \frac{\sigma_x}{B}} \tag{4-40}$$

Note that, in either case, the final "fractional error" (σ_u/u or σ_v/v) is the same as the original fractional error (σ_x/x). As we would expect intuitively, multiplying or dividing a value by a constant does not change its relative error.

A familiar example of the above case is the calculation of a counting rate. If x counts are recorded over a time t, then

$$\text{counting rate} \equiv r = \frac{x}{t}.$$

The usual assumption is that the time is measured with very small uncertainty, so that t can be considered a constant. Then Eq. 4-40 can be used to calculate the expected standard deviation in r corresponding to the known standard deviation in the number of counts x.

As an example, suppose

$$x = 1120 \text{ counts and } t = 5 \text{ seconds}$$

Then

$$r = \frac{1120}{5s} = 224 s^{-1}$$

The associated standard deviation is

$$\sigma_r = \frac{\sigma_x}{t} = \frac{\sqrt{1120}}{5s} = 6.7 s^{-1}$$

Therefore, the counting rate is

$$r = 224 \pm 6.7 \text{ counts per second}$$

Case 3. Multiplication or Division of Counts

For the case

$$u = xy, \qquad \frac{\partial u}{\partial x} = y, \qquad \frac{\partial u}{\partial y} = x$$

$$\sigma_u^2 = y^2\sigma_x^2 + x^2\sigma_y^2$$

Dividing both sides by $u^2 = x^2y^2$

$$\boxed{\left(\frac{\sigma_u}{u}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2} \qquad (4\text{-}41)$$

Similarly, if

$$u = \frac{x}{y}, \qquad \frac{\partial u}{\partial x} = \frac{1}{y}, \qquad \frac{\partial u}{\partial y} = -\frac{x}{y^2}$$

$$\sigma_u^2 = \left(\frac{1}{y}\right)^2\sigma_x^2 + \left(-\frac{x}{y^2}\right)^2\sigma_y^2$$

Again, dividing both sides by $u^2 = \frac{x^2}{y^2}$

$$\boxed{\left(\frac{\sigma_u}{u}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2} \qquad (4\text{-}41')$$

Thus, for either $u = xy$ or $u = x/y$, the *fractional errors* in x and y (σ_x/x and σ_y/y) combine in quadrature sum to give the *fractional error* in u.

As an example, suppose we wish to calculate the ratio of two source activities from independent counts taken for equal counting times (background is neglected). Assume

$$\text{Counts from source } ① \equiv N_1 = 16265$$
$$\text{Counts from source } ② \equiv N_2 = 8192$$
$$\text{Activity ratio: } R \equiv \frac{N_1}{N_2} = \frac{16265}{8192} = 1.985$$

From Eq. 4-41′

$$\left(\frac{\sigma_R}{R}\right)^2 = \left(\frac{\sigma_{N_1}}{N_1}\right)^2 + \left(\frac{\sigma_{N_2}}{N_2}\right)^2 = \frac{N_1}{N_1^2} + \frac{N_2}{N_2^2}$$
$$= 1.835 \times 10^{-4}$$
$$\frac{\sigma_R}{R} = 0.0135$$

and multiplying by the value of R

$$\sigma_R = 0.027$$

Therefore, the reported result would be:

$$R = 1.985 \pm 0.027$$

Case 4. Mean Value of Multiple Independent Counts

Suppose we have recorded N repeated counts from the same source for equal counting times. Let the results of these multiple counts be designated $x_1, x_2, \ldots x_N$ and their sum be Σ. Then

$$\Sigma = x_1 + x_2 + \ldots x_N$$

If we formally apply the error propagation formula (Eq. 4-37) to find the expected error in Σ, we find $\partial\Sigma/\partial x_i = 1$ for all independent counts x_i, and therefore,

$$\sigma_\Sigma^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + \ldots \sigma_{x_N}^2$$

But because $\sigma_{x_i} = \sqrt{x_i}$ for each independent count,

$$\overset{2}{\Sigma} = x_1 + x_2 + \ldots x_N = \Sigma$$

$$\sigma_\Sigma = \sqrt{\Sigma} \tag{4-42}$$

This result shows that the standard deviation expected for the sum of all the counts is the same as if the measurement had been carried out by performing a single count, extending over the entire period represented by all the independent counts.

Now if we proceed to calculate a mean value from these N independent measurements,

$$\bar{x} = \frac{\Sigma}{N} \tag{4-43}$$

Equation 4-43 is an example of dividing an error-associated quantity (Σ) by a constant (N). Therefore, Eq. 4-40 applies and the expected standard deviation of this mean value is given by

$$\sigma_{\bar{x}} = \sigma_\Sigma / N = \sqrt{\Sigma}\ / N = \sqrt{N\bar{x}}\ / N$$

$$\boxed{\sigma_{\bar{x}} = \sqrt{\frac{\bar{x}}{N}}} \tag{4-44}$$

Note that the expected standard deviation of any single measurement x_i is

$$\sigma_{x_i} = \sqrt{x_i}$$

Because any typical count will not differ greatly from the mean, $x_i \cong \bar{x}$, and we therefore conclude that the mean value based on N independent counts will have an expected error which is smaller by a factor $\sqrt{N}$ compared with any single measurement on which the mean is based. A general conclusion is that, if we wish to improve the statistical precision of a given measurement by a factor of two, we must invest four times the initial counting time.

Case 5. Combination of Independent Measurements with Unequal Errors

If N independent measurements of the same quantity have been carried out and they do not all have nearly the same associated precision, then a simple average (as discussed in Case 4) no longer is the optimal way to calculate a single "best value." We will instead want to give more weight to those measurements with small values for σ_{x_i} (the standard deviation associated with x_i) and less weight to measurements for which this estimated error is large.

Let each individual measurement x_i be given a weighting factor a_i and the "best value" $\langle x \rangle$ computed from the linear combination

$$\langle x \rangle = \frac{\sum_{i=1}^{N} a_i x_i}{\sum_{i=1}^{N} a_i} \tag{4-45}$$

We will now seek a criterion by which the weighting factors a_i should be chosen in order to minimize the expected error in $\langle x \rangle$.

For brevity, we will write

$$\alpha \equiv \sum_{i=1}^{N} a_i$$

so that

$$\langle x \rangle = \frac{1}{\alpha} \sum_{i=1}^{N} a_i x_i$$

Now apply the error propagation formula (Eq. 4-37) to this case:

$$\sigma_{\langle x \rangle}^2 = \sum_{i=1}^{N} \left(\frac{\partial \langle x \rangle}{\partial x_i} \right)^2 \sigma_{x_i}^2$$

$$= \sum_{i=1}^{N} \left(\frac{a_i}{\alpha} \right)^2 \sigma_{x_i}^2$$

$$= \frac{1}{\alpha^2} \sum_{i=1}^{N} a_i^2 \sigma_{x_i}^2$$

$$\sigma_{\langle x \rangle}^2 = \beta / \alpha^2 \tag{4-46}$$

where

$$\beta \equiv \sum_{i=1}^{N} a_i^2 \sigma_{x_i}^2$$

In order to minimize $\sigma_{\langle x \rangle}$, we must minimize $\sigma_{\langle x \rangle}^2$ from Eq. 4-46 with respect to a typical weighting factor a_j

$$0 = \frac{\partial \sigma_{\langle x \rangle}^2}{\partial a_j} = \frac{\alpha^2 \dfrac{\partial \beta}{\partial a_j} - 2\alpha\beta \dfrac{\partial \alpha}{\partial a_j}}{\alpha^4} \tag{4-47}$$

Note that

$$\frac{\partial \alpha}{\partial a_j} = 1 \qquad \frac{\partial \beta}{\partial a_j} = 2a_j \sigma_{x_j}^2$$

Putting these results into Eq. 4-47

$$\frac{1}{\alpha^4}\left(2\alpha^2 a_j \sigma_{x_j}^2 - 2\alpha\beta\right) = 0$$

and solving for a_j

$$a_j = \frac{\beta}{\alpha} \cdot \frac{1}{\sigma_{x_j}^2} \tag{4-48}$$

If we choose to normalize the weighting coefficients

$$\sum_{i=1}^{N} a_i \equiv \alpha = 1$$

$$a_j = \frac{\beta}{\sigma_{x_j}^2}$$

Putting this into the definition of β

$$\beta = \sum_{i=1}^{N} a_i^2 \sigma_{x_i}^2 = \sum_{i=1}^{N} \left(\frac{\beta}{\sigma_{x_i}^2}\right)^2 \sigma_{x_i}^2$$

or

$$\beta = \left[\sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2}\right]^{-1} \tag{4-49}$$

Therefore, the proper choice for the normalized weighting coefficient for x_j is

$$\boxed{a_j = \frac{1}{\sigma_{x_j}^2}\left[\sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2}\right]^{-1}} \tag{4-50}$$

We therefore see that *each data point should be weighted inversely as the square of its own error.*

Assuming that this optimal weighting is followed, what will be the resultant (minimum) error in $\langle x \rangle$? Because we have chosen $\alpha = 1$ for normalization, Eq. 4-46 becomes

$$\sigma_{\langle x \rangle}^2 = \beta$$

In the case of optimal weighting, β is given by Eq. 4-49. Therefore,

$$\boxed{\frac{1}{\sigma_{\langle x \rangle}^2} = \sum_{i=1}^{N} \frac{1}{\sigma_{x_i}^2}} \tag{4-51}$$

From Eq. 4-51, the expected standard deviation $\sigma_{\langle x \rangle}$ can be calculated from the standard deviations σ_{x_i} associated with each individual measurement.

V. OPTIMIZATION OF COUNTING EXPERIMENTS

The principle of error propagation can be applied in the design of counting experiments to minimize the associated statistical uncertainty. To illustrate, consider the simple case of measurement of the net counting rate from a long-lived radioactive source in the presence of a steady-state background. Define the following:

$S \equiv$ counting rate due to the source alone without background

$B \equiv$ counting rate due to background

The measurement of S is normally carried out by counting the source plus background (at an average rate of $S+B$) for a time T_{S+B}, and then counting background alone for a time T_B. The net rate due to the source alone is then

$$S = \frac{N_1}{T_{S+B}} - \frac{N_2}{T_B} \tag{4-52}$$

where N_1 and N_2 are the total counts in each measurement.

Applying the results of error propagation analysis to Eq. 4-52

$$\sigma_S = \left[\left(\frac{\sigma_{N_1}}{T_{S+B}} \right)^2 + \left(\frac{\sigma_{N_2}}{T_B} \right)^2 \right]^{1/2}$$

$$\sigma_S = \left[\frac{N_1}{T_{S+B}^2} + \frac{N_2}{T_B^2} \right]^{1/2}$$

$$\sigma_S = \left[\frac{S+B}{T_{S+B}} + \frac{B}{T_B} \right]^{1/2} \tag{4-53}$$

If we now assume that a fixed total time $T = T_{S+B} + T_B$ is available to carry out both measurements, the above uncertainty can be minimized by optimally choosing the fraction of T allocated to T_{S+B} (or T_B). We square Eq. 4-53 and differentiate

$$2\sigma_S \, d\sigma_S = -\frac{S+B}{T_{S+B}^2} dT_{S+B} - \frac{B}{T_B^2} dT_B$$

and set $d\sigma_S = 0$ to find the optimum condition. Also, because T is a constant, $dT_{S+B} + dT_B = 0$. The optimum division of time is then obtained by meeting the condition

$$\left. \frac{T_{S+B}}{T_B} \right]_{\text{opt}} = \sqrt{\frac{S+B}{B}} \tag{4-54}$$

A figure of merit which can be used to characterize this type of counting experiment is the inverse of the total time, or $1/T$, required to determine S to within a given statistical accuracy. If certain parameters of the experiment (such as detector size, pulse acceptance criteria, etc.) can be varied, the optimal choice should correspond to maximizing this figure of merit.

In the following analysis, we will assume that the optimal division of counting times given by Eq. 4-54 is chosen. Then we can combine Eqs. 4-53 and 4-54 to obtain an expression for the figure of merit in terms of the fractional standard deviation of the source rate, defined as $\epsilon \equiv \sigma_S / S$

$$\frac{1}{T} = \epsilon^2 \frac{S^2}{(\sqrt{S+B} + \sqrt{B})^2} \tag{4-55}$$

It is instructive to examine two extreme cases in the application of this result. If the source-induced rate is much greater than the background, $S \gg B$ and Eq. 4-55 reduces to

$$\frac{1}{T} \cong \epsilon^2 S \tag{4-56}$$

In this limit, the statistical influence of background is negligible. The figure of merit $1/T$ is maximized simply by choosing all experiment parameters to maximize S, or the rate due to the source alone.

The opposite extreme of a small source rate in a much larger background ($S \ll B$) is typical of low-level radioactivity measurements. In this case, Eq. 4-55 reduces to

$$\frac{1}{T} \cong \epsilon^2 \frac{S^2}{4B} \tag{4-57}$$

For such applications, the figure of merit is maximized by choosing experimental conditions so that the ratio S^2/B is maximized. As an example of the application of Eq. 4-57, assume that changing the detector configuration in a low-level counting experiment increases the rate due to the source alone by a factor of 1.5, but also increases the background by a factor of 2.0. The ratio S^2/B is then $(1.5)^2/2.0 = 1.125$ times its former value. Because this ratio exceeds unity, the change will slightly improve the overall statistical accuracy of the net source rate determination if the total measurement time is held constant.

VI. DISTRIBUTION OF TIME INTERVALS

The time intervals separating random events are often of practical interest in nuclear measurements. We will present some results which apply to any random process characterized by a constant probability of occurrence per unit time. In most cases, these results will adequately describe the behavior of a radiation detector undergoing irradiation by a steady-state or long-lived radiation source.

In the following discussion, r will represent the average rate at which events are occurring. It follows that rdt is the differential probability that an event will take place in the differential time increment dt. For a radiation detector with unity efficiency counting a single radioisotope

$$r = \left| \frac{dN}{dt} \right| = \lambda N$$

where N is the number of radioactive nuclei and λ is their decay constant.

A. Intervals Between Successive Events

In order to derive a distribution function to describe the time intervals between adjacent random events, first assume that an event has occurred at time $t = 0$. What is the differential probability that the next event will take place within a

differential time dt after a time interval of length t? Two independent processes must take place: No events may occur within the time interval from 0 to t, but an event must take place in the next differential time increment dt. The overall probability will then be given by the product of the probabilities characterizing the two processes, or

Probability of next event taking place in dt after delay of t	=	Probability of *no* events during time from 0 to t	×	Probability of an event during dt
$I_1(t)\,dt$	=	$P(0)$	×	$r\,dt$

(4-58)

The first factor on the right-hand side follows directly from the earlier discussion of the Poisson distribution. We seek the probability that no events will be recorded over an interval of length t for which the average number of recorded events should be rt. From Eq. 4-24

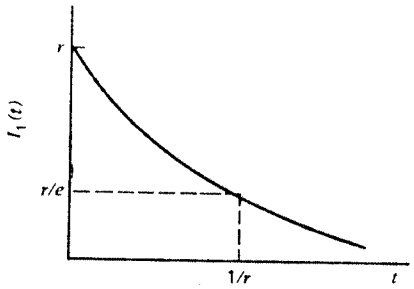
$$P(0)=\frac{(rt)^0 e^{-rt}}{0!}$$

$$P(0)=e^{-rt} \tag{4-59}$$

Substituting Eq. 4-59 into Eq. 4-58 leads to

$$\boxed{I_1(t)\,dt=re^{-rt}\,dt} \tag{4-60}$$

$I_1(t)$ is now the distribution function for intervals between adjacent random events. The plot below shows the simple exponential shape of this distribution.



Note that the *most probable* interval is zero. The *average* interval length is calculated by

$$\bar{t}=\frac{\int_0^\infty tI_1(t)\,dt}{\int_0^\infty I_1(t)\,dt}=\frac{\int_0^\infty te^{-rt}\,dt}{\int_0^\infty e^{-rt}\,dt}=\frac{1}{r} \tag{4-61}$$

which is the expected result.

In some radiation applications, the counting rate is low enough so that each individual count can be visually observed as the data are being collected. Experienced observers soon learn what a true exponential interval distribution "looks like" and occasionally can spot a malfunctioning detector by noting a deviation from an expected random input signal.

B. Intervals Between Scaled Events

Event rates are often high enough to necessitate the use of a scaler before the output of the detector can be recorded. The scaler serves to reduce the apparent rate by producing an output pulse only when N input pulses have been accumulated.

A general form for the distribution which describes scaled intervals can be derived using arguments parallel to those given earlier for unscaled intervals. Again, two independent processes must occur: A time interval of length t must be observed over which exactly $(N-1)$ events are presented to the scaler, and an additional event must occur in the increment dt following this time interval. Under these conditions, a scaled interval of length within dt about t will take place. The parallel expression to Eq. 4-58 then becomes

$$I_N(t)\,dt=P(N-1)r\,dt \tag{4-62}$$

Again using the Poisson form for $P(N-1)$, Eq. 4-62 becomes

$$\boxed{I_N(t)\,dt=\frac{(rt)^{N-1}e^{-rt}}{(N-1)!}r\,dt} \tag{4-63}$$

$I_N(t)$ is the interval distribution for N-scaled intervals. A plot is given in Fig. 4-14 for various scaling factors and shows the more uniform intervals that accompany larger values of N. The average interval is

$$\bar{t}=\frac{\int_0^\infty tI_N(t)\,dt}{\int_0^\infty I_N(t)\,dt}=\frac{N}{r} \tag{4-64}$$

whereas the most probable interval is evaluated by setting

$$\frac{dI_N(t)}{dt}=0$$

and leads to

$$t\Big]_{\substack{\text{most}\\ \text{probable}}}=\frac{N-1}{r} \tag{4-65}$$

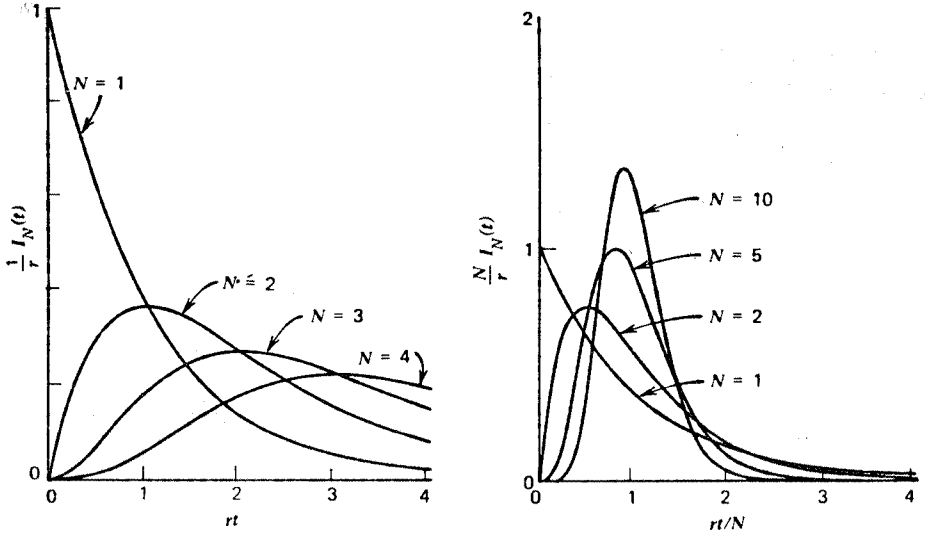


FIGURE 4-14. Graphical representations of the scaled interval distribution $I_N(t)$. Part (a) shows four distributions for scaling factors of 1, 2, 3, and 4. Part (b) plots interval distributions for $N=1$ through $N=10$ normalized to the same average interval N/r.

PROBLEMS

4-1. A source is counted for 1 minute and gives 561 counts. The source is removed and a 1-minute background count gives 410 counts. What is the net count due to the source alone and its associated standard deviation?

4-2. A 10-minute count of a source + background gives a total of 846 counts. Background alone counted for 10 minutes gives a total of 73 counts. What is the net counting rate due to source alone, and what is its associated standard deviation?

4-3. The measurement described in Problem 4-2 is to be repeated, but in this case the available 20 minutes is to be subdivided optimally between the two separate counts. Find the optimal allocation of time that minimizes the expected standard deviation in the net source counting rate. By what factor has the expected statistical error been reduced from the situation of Problem 4-2?

4-4. A flow counter shows an average background rate of 2.87 counts per minute. What is the probability that a given two-minute count will contain:

(a) exactly five counts
(b) at least one count.

What length of counting time is required to insure with >99 percent probability that at least one count is recorded?

4-5. The following data are obtained from sources A and B of the same isotope:

		Timing Period
Source A + background	251 counts	5 minutes
Source B + background	717 counts	2 minutes
Background	51 counts	10 minutes

What is the ratio of the activity of source B to source A, and what is the percent standard deviation in this ratio?

4-6. The background count from a detector was measured to be 845 over a 30-minute period. A source to be measured increases the total counting rate to about 80 counts per minute. Estimate the time the source should be counted to determine the counting rate due to the source alone to within a fractional standard deviation of 3 percent.

4-7. Thirty different students have measured the background counting rate with the same apparatus. Each used the same procedure, consisting of recording the number of counts in five 1-minute intervals, and taking their average. A set of numbers from a typical student is shown below:

$$\begin{aligned} 25 &= \text{count in first minute} \\ 35 &= \text{count in second minute} \\ 30 &= \text{count in third minute} \\ 23 &= \text{count in fourth minute} \\ 27 &= \text{count in fifth minute} \\ \text{total} &= 140 \end{aligned}$$

$$\text{mean} = \frac{140}{5} = 28.0 \frac{\text{counts}}{\text{minute}}$$

(a) Does this data seem reasonable assuming all the fluctuations are statistical? Substantiate your conclusion quantitatively.
(b) Based on the above data, what is the expected standard deviation of the mean?
(c) Estimate the sample variance of the 30 numbers representing a similar calculation of the mean background rate by each of the 30 students.
(d) Again assuming only statistical variations, estimate the standard deviation of the final answer for the mean obtained by averaging all 30 independent values.

4-8. The following set of 25 counts was recorded under identical detector conditions and counting times. Apply the Chi-squared test to determine

whether the observed fluctuations are consistent with expectations from Poisson statistics.

3626	3711	3677	3678	3465
3731	3617	3630	3624	3574
3572	3572	3615	3652	3601
3689	3578	3605	3595	3540
3625	3569	3591	3636	3629

4-9. An average of five sequential 2-minute counts of a constant source by Lab Group A gave a resulting value of 2162.4 counts per minute. Lab Group B then used the same source and detector in identical conditions and arrived at a value of 2081.5 counts per minute based on four sequential 5-minute counts. Is the difference between these two results statistically significant?

4-10. You are asked to calibrate the activity of a Cs-137 gamma ray source by comparison with a standard Cs-137 reference source of approximately the same activity. The standard source has a quoted activity of 3.50 ± 0.05 microcuries ($\pm$ one standard deviation) and either source alone gives rise to a counting rate of about 1000 per second in the available counter. Background rates are negligible.

Assuming that each source is counted separately for equal counting times, how much *total* time will be required in order to determine the unknown activity to within a 2 percent expected standard deviation?

4-11. A particular counting system has an inherent average background rate of 50 counts per minute. A decaying radioisotope source was introduced and a 10-minute count showed a total of 1683 counts. After a delay of 24 hours, the 10-minute count was repeated, this time giving a total of 914 counts.

(a) What is the half-life of the source?

(b) What is the expected standard deviation of the half-life value due to counting statistics?

4-12. An engine wear test is to be carried out in which the weight of radioactive piston ring particles in an oil sample is to be determined. A sample of the used oil gives 13,834 counts over a 3-minute period. A standard has been prepared using exactly 100 μg of the same activity material which gives 91,396 counts over a 10-minute period. Background for the detector has been determined to be 281 counts per minute, measured over a very long counting period (~24 hours). Find the weight of particles in the sample and its expected fractional standard deviation.

4-13. The decay constant λ of radioisotope sample is to be determined by counting in a detector system with negligible background. An approximate value λ' is already known. The procedure will be to count for a short time τ at $t=0$, wait a time Δt, and then count again for the same time τ. Assuming that $\tau \ll 1/\lambda'$, what value of the waiting time Δt will minimize the expected statistical error in the value of λ derived from these measurements?

4-14. The thickness of nominal 1 cm sheet aluminum is to be monitored by noting the attenuation of a gamma ray parallel beam passing perpendicularly through the sheet. The source and detector are well shielded, so background and scattering into the detector are negligible. Any given sample will spend 1 second in the beam. The detector counting rate with no sheet in place has a mean value (measured over a long time) of 10,000 per second.

(a) Find the optimum value of the linear attenuation coefficient μ which will minimize the uncertainty in the derived sheet thickness value due to statistical fluctuations. (What is the corresponding gamma ray energy?)

(b) What is the lowest attainable fractional standard deviation under these conditions?

REFERENCES

1. R. D. Evans, *The Atomic Nucleus*, McGraw-Hill, New York (1955), Chapters 26, 27, and 28.
2. W. Feller, *An Introduction to Probability Theory and Its Applications*, 2nd Edition, John Wiley, New York (1957).
3. P. R. Bevington, *Data Reduction and Error Analysis for the Physical Sciences*, McGraw-Hill, New York (1969).