Lista de exercícios 8

O operador de Wigner para a translação do centro de massa atômico é definido como

$$f(\mathbf{r},\mathbf{p}) \equiv \frac{1}{(2\pi\hbar)^3}\int_{V_\infty} d^3s \left|\mathbf{r}-\frac{1}{2}\mathbf{s}\right\rangle\left\langle\mathbf{r}+\frac{1}{2}\mathbf{s}\right| \exp\left(-i\frac{\mathbf{p}\cdot\mathbf{s}}{\hbar}\right).$$

Primeira questão

Mostre que

$$\frac{\mathbf{P}_{CM}^2}{2M} = \int d^3r' \int d^3p' \frac{\mathbf{p}'^2}{2M} f(\mathbf{r}',\mathbf{p}').$$

Segunda questão

Mostre que

$$\Omega_L(\mathbf{R}_{CM}) = \int d^3r' \int d^3p' \Omega_L(\mathbf{r}') f(\mathbf{r}',\mathbf{p}').$$

Terceira questão

Mostre que

$$\begin{aligned}[f(\mathbf{r},\mathbf{p}), f(\mathbf{r}',\mathbf{p}')] &= \frac{-2i}{(\pi\hbar)^6}\int d^3r'' \int d^3p'' f(\mathbf{r}'',\mathbf{p}'') \\ &\times \sin\left\{\frac{2}{\hbar}\left[(\mathbf{p}-\mathbf{p}')\cdot\left(\frac{\mathbf{r}'+\mathbf{r}}{2}-\mathbf{r}''\right)-(\mathbf{r}-\mathbf{r}')\cdot\left(\frac{\mathbf{p}'+\mathbf{p}}{2}-\mathbf{p}''\right)\right]\right\}.\end{aligned}$$

Quarta questão

Mostre que

$$\begin{aligned}\int d^3p' [f(\mathbf{r},\mathbf{p}), f(\mathbf{r}',\mathbf{p}')] &= \frac{-2i}{(\pi\hbar)^3}\int d^3p'' \int d^3r'' \delta^{(3)}(\mathbf{r}-\mathbf{r}'') f(\mathbf{r}'',\mathbf{p}'') \\ &\times \mathrm{Im}\left\{\exp\left[-\frac{2}{\hbar}i(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{p}-\mathbf{p}'')\right]\right\} \\ &= \frac{-2i}{(\pi\hbar)^3}\int d^3p'' f(\mathbf{r},\mathbf{p}'')\,\mathrm{Im}\left\{\exp\left[-\frac{2}{\hbar}i(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{p}-\mathbf{p}'')\right]\right\}.\end{aligned}$$

Quinta questão

Mostre que

$$\left[f(\mathbf{r},\mathbf{p}), \int d^3r' d^3p' \mathbb{V}(\mathbf{r}',t) f(\mathbf{r}',\mathbf{p}')\right] = \int d^3p'' \mathbb{J}(\mathbf{r},\mathbf{p}-\mathbf{p}'',t) f(\mathbf{r},\mathbf{p}''),$$

where we have defined

$$
\begin{aligned}
\mathbb{J}(\mathbf{r},\mathbf{p},t) &\equiv \frac{-2i}{(\pi\hbar)^3}\int d^3r'\mathbb{V}(\mathbf{r}',t)\,\mathrm{Im}\left\{\exp\left[-\frac{2}{\hbar}i(\mathbf{r}-\mathbf{r}')\cdot\mathbf{p}\right]\right\} \\
&= \frac{-1}{(\pi\hbar)^3}\int d^3r'\mathbb{V}(\mathbf{r}',t)\exp\left[-\frac{2}{\hbar}i(\mathbf{r}-\mathbf{r}')\cdot\mathbf{p}\right] \\
&\quad +\frac{1}{(\pi\hbar)^3}\int d^3r'\mathbb{V}(\mathbf{r}',t)\exp\left[\frac{2}{\hbar}i(\mathbf{r}-\mathbf{r}')\cdot\mathbf{p}\right],
\end{aligned}
$$

Onde $\mathbb{V}(\mathbf{r},t)$ é um operador que não atua no centro de massa, mas depende de $\mathbf{r}$ e t.