Pure decoherence of a qubit

We are going to start by writing the Hamiltonian of a single qubit:

$$\begin{aligned} H &= H_0 + \hbar \sum_s \omega_s b_s^\dagger b_s \\ &\quad + \hbar \sigma_z \sum_s \left(g_s b_s + g_s^* b_s^\dagger \right). \end{aligned}$$

Here we are going to assume that

$$[H_0, \sigma_z] = 0.$$

Therefore, in the interaction picture we have

$$H_I(t) = \hbar \sigma_z \sum_s \left[g_s \exp(-i\omega_s t) b_s + g_s^* b_s^\dagger \exp(i\omega_s t) \right]$$

and the evolution operator satisfies

$$i\hbar \frac{d}{dt} U_I(t) = H_I(t) U_I(t).$$

Now we are going to prove that the evolution operator is given by

$$U_I(t) = \exp[C(t)] \exp[-\sigma_z B(t)] \exp[\sigma_z A(t)],$$

where

$$C(t) \equiv \sum_s |g_s|^2 \frac{i\omega_s t + \exp(-i\omega_s t) - 1}{\omega_s^2},$$

$$B(t) \equiv \sum_{s'} g_{s'}^* f_{s'}^*(t) b_{s'}^\dagger,$$

$$A(t) \equiv \sum_{s'} g_{s'} f_{s'}(t) b_{s'},$$

and

$$f_s(t) \equiv \frac{\exp(-i\omega_s t) - 1}{\omega_s}.$$

Hence, after doing all the calculations, we obtain, by direct differentiation and boson operator algebra,

$$i\hbar \frac{d}{dt} U_I(t) = \hbar \sigma_z \left[\sum_s g_s^* b_s^\dagger \exp(i\omega_s t) + g_s b_s \exp(-i\omega_s t) \right] U_I(t),$$

that is,

$$i\hbar \frac{d}{dt} U_I(t) = H_I(t) U_I(t).$$

Since now we have the evolution operator, we need to calculate the evolution of the qubit initial state. Let us assume that, at $t = 0$, the qubit is in the state

$$|\psi(0)\rangle = c_0|0\rangle + c_1|1\rangle,$$

with

$$|c_0|^2 + |c_1|^2 = 1.$$

The initial qubit density operator is thus

$$\begin{aligned}\rho_S(0) &= |\psi(0)\rangle\langle\psi(0)| \\ &= |c_0|^2|0\rangle\langle 0| + c_0c_1^*|0\rangle\langle 1| \\ &\quad + c_0^*c_1|1\rangle\langle 0| + |c_1|^2|1\rangle\langle 1|.\end{aligned}$$

Let us take a thermal state for the bath:

$$\rho_B(0) = \frac{1}{Z}\exp\left(-\beta\hbar\sum_s \omega_s b_s^\dagger b_s\right),$$

where

$$\beta \equiv \frac{1}{k_B T}.$$

Thus, at $t = 0$, we have

$$\rho_I(0) = \rho_S(0)\rho_B(0).$$

We know that

$$\begin{aligned}\rho_I(t) &= U_I(t)\rho_S(0)\rho_B(0)U_I^\dagger(t) \\ &= |c_0|^2 U_I(t)|0\rangle\langle 0|\rho_B(0)U_I^\dagger(t) \\ &\quad + c_0c_1^* U_I(t)|0\rangle\langle 1|\rho_B(0)U_I^\dagger(t) \\ &\quad + c_0^*c_1 U_I(t)|1\rangle\langle 0|\rho_B(0)U_I^\dagger(t) \\ &\quad + |c_1|^2 U_I(t)|1\rangle\langle 1|\rho_B(0)U_I^\dagger(t),\end{aligned}$$

that is, using a more appealing notation (at least to me),

$$\begin{aligned}\rho_I(t) &= |c_0|^2 U_I(t)|0\rangle\rho_B(0)\langle 0|U_I^\dagger(t) \\ &\quad + c_0c_1^* U_I(t)|0\rangle\rho_B(0)\langle 1|U_I^\dagger(t) \\ &\quad + c_0^*c_1 U_I(t)|1\rangle\rho_B(0)\langle 0|U_I^\dagger(t) \\ &\quad + |c_1|^2 U_I(t)|1\rangle\rho_B(0)\langle 1|U_I^\dagger(t).\end{aligned}$$

Some more algebra and we obtain

$$\begin{aligned}U_I(t)|0\rangle\rho_B(0)\langle 0|U_I^\dagger(t) &= |0\rangle\langle 0|\exp[C^*(t)]\exp[C(t)]\exp[-B(t)] \\ &\quad \times\exp[A(t)]\rho_B(0)\exp\left[A^\dagger(t)\right]\exp\left[-B^\dagger(t)\right],\end{aligned}$$

$$\begin{aligned}U_I(t)|0\rangle\rho_B(0)\langle 1|U_I^\dagger(t) &= |0\rangle\langle 1|\exp[C^*(t)]\exp[C(t)]\exp[-B(t)] \\ &\quad\times\exp[A(t)]\rho_B(0)\exp\left[-A^\dagger(t)\right]\exp\left[B^\dagger(t)\right],\end{aligned}$$

$$\begin{aligned}U_I(t)|1\rangle\rho_B(0)\langle 0|U_I^\dagger(t) &= |1\rangle\langle 0|\exp[C^*(t)]\exp[C(t)]\exp[B(t)] \\ &\quad\times\exp[-A(t)]\rho_B(0)\exp\left[A^\dagger(t)\right]\exp\left[-B^\dagger(t)\right],\end{aligned}$$

and

$$\begin{aligned}U_I(t)|1\rangle\rho_B(0)\langle 1|U_I^\dagger(t) &= |1\rangle\langle 1|\exp[C^*(t)]\exp[C(t)]\exp[B(t)] \\ &\quad\times\exp[-A(t)]\rho_B(0)\exp\left[-A^\dagger(t)\right]\exp\left[B^\dagger(t)\right].\end{aligned}$$

But,

$$C(t) = \sum_s |g_s|^2\,\frac{i\omega_s t+\exp(-i\omega_s t)-1}{\omega_s^2}$$

and, hence,

$$C^*(t) = \sum_s |g_s|^2\,\frac{-i\omega_s t+\exp(i\omega_s t)-1}{\omega_s^2},$$

so that

$$C^*(t)+C(t) = 2\sum_s |g_s|^2\,\frac{\cos(\omega_s t)-1}{\omega_s^2}.$$

We need now calculate $\mathrm{Tr}_B[\rho_I(t)]$. Then, let us use the following coherent-state expansion of the bath state:

$$\rho_B(0) = \prod_s \frac{1}{\langle n_s\rangle}\frac{1}{\pi}\int d^2\alpha_s\,\exp\left(-\frac{|\alpha_s|^2}{\langle n_s\rangle}\right)|\alpha_s\rangle\langle\alpha_s|,$$

where

$$\langle n_s\rangle = \frac{1}{\exp(\beta\hbar\omega_s)-1}.$$

It immediately follows that

$$\begin{aligned}\mathrm{Tr}_B\left[U_I(t)|0\rangle\rho_B(0)\langle 0|U_I^\dagger(t)\right] &= |0\rangle\langle 0|\exp[C^*(t)]\exp[C(t)] \\ &\quad\times\prod_s\exp\left[|g_s|^2|f_s(t)|^2\right] \\ &= |0\rangle\langle 0|\exp[C^*(t)]\exp[C(t)] \\ &\quad\times\exp[-C^*(t)-C(t)] \\ &= |0\rangle\langle 0|\end{aligned}$$

and

$$\mathrm{Tr}_B\left[U_I(t)|1\rangle\rho_B(0)\langle 1|U_I^\dagger(t)\right] = |1\rangle\langle 1|,$$

as expected. We also calculate:

$$\begin{aligned}\mathrm{Tr}_B\left[U_I(t)\left|0\right\rangle\rho_B(0)\left\langle 1\right|U_I^\dagger(t)\right] &= \left|0\right\rangle\left\langle 1\right|\exp\left[C^*(t)\right]\exp\left[C(t)\right]\mathrm{Tr}_B\left\{\exp\left[-B(t)\right]\right.\\ &\quad\left.\times\exp\left[A(t)\right]\rho_B(0)\exp\left[-A^\dagger(t)\right]\exp\left[B^\dagger(t)\right]\right\}.\end{aligned}$$

With some more algebra, we obtain

$$\begin{aligned}\mathrm{Tr}_B\left[U_I(t)\left|0\right\rangle\rho_B(0)\left\langle 1\right|U_I^\dagger(t)\right] &= \left|0\right\rangle\left\langle 1\right|\exp\left[4\sum_s\left|g_s\right|^2\frac{\cos(\omega_s t)-1}{\omega_s^2}\right]\\ &\quad\times\prod_s\frac{1}{\left\langle n_s\right\rangle}\frac{1}{\pi}\int d^2\alpha_s\,\exp\left(-\frac{\left|\alpha_s\right|^2}{\left\langle n_s\right\rangle}\right)\\ &\quad\times\exp\left[2g_s f_s(t)\,\alpha_s-2g_s^*f_s^*(t)\,\alpha_s^*\right].\end{aligned}$$

Calculating the integral finally gives

$$\begin{aligned}\mathrm{Tr}_B\left[U_I(t)\left|0\right\rangle\rho_B(0)\left\langle 1\right|U_I^\dagger(t)\right] &= \left|0\right\rangle\left\langle 1\right|\exp\left[4\sum_s\left|g_s\right|^2\frac{\cos(\omega_s t)-1}{\omega_s^2}\right]\\ &\quad\times\exp\left[-4\sum_s\left\langle n_s\right\rangle\left|g_s\right|^2\left|f_s(t)\right|^2\right]\\ &= \left|0\right\rangle\left\langle 1\right|\exp\left[4\sum_s\left|g_s\right|^2\frac{\cos(\omega_s t)-1}{\omega_s^2}\right]\\ &\quad\times\exp\left[-8\sum_s\left\langle n_s\right\rangle\left|g_s\right|^2\frac{1-\cos(\omega_s t)}{\omega_s^2}\right].\end{aligned}$$

We now take the continuum limit using an ohmic spectral distribution:

$$\begin{aligned}\sum_s\left|g_s\right|^2\frac{\cos(\omega_s t)-1}{\omega_s^2} &= \int_0^\infty d\omega\,\sum_s\left|g_s\right|^2\delta(\omega-\omega_s)\,\frac{\cos(\omega t)-1}{\omega^2}\\ &= \int_0^\infty d\omega\,J(\omega)\,\frac{\cos(\omega t)-1}{\omega^2}\\ &= \int_0^\infty d\omega\,\eta\omega\exp\left(-\frac{\omega}{\omega_c}\right)\frac{\cos(\omega t)-1}{\omega^2}\\ &= \eta\int_0^\infty d\omega\,\exp\left(-\frac{\omega}{\omega_c}\right)\frac{\cos(\omega t)-1}{\omega}\\ &= -\frac{\eta}{2}\ln\left(1+\omega_c^2t^2\right).\end{aligned}$$

where we have used

$$J(\omega)\equiv\sum_s\left|g_s\right|^2\delta(\omega-\omega_s)$$

and

$$J(\omega)\equiv\eta\omega\exp\left(-\frac{\omega}{\omega_c}\right).$$

The reduced density operator of the qubit is, therefore, given by

$$\begin{aligned}
\mathrm{Tr}_B\left[\rho_I(t)\right] &= \left|c_0\right|^2 |0\rangle\langle 0| \\
&+ \frac{c_0 c_1^* |0\rangle\langle 1|}{\left(1+\omega_c^2 t^2\right)^{2\eta}} \exp\left[8\sum_s \langle n_s\rangle \left|g_s\right|^2 \frac{\cos(\omega_s t)-1}{\omega_s^2}\right] \\
&+ \frac{c_0^* c_1 |1\rangle\langle 0|}{\left(1+\omega_c^2 t^2\right)^{2\eta}} \exp\left[8\sum_s \langle n_s\rangle \left|g_s\right|^2 \frac{\cos(\omega_s t)-1}{\omega_s^2}\right] \\
&+ \left|c_1\right|^2 |1\rangle\langle 1| .
\end{aligned}$$

We see, therefore, that we need to calculate the quantity

$$\begin{aligned}
\mathscr{I}(t) &\equiv 8\sum_s \langle n_s\rangle \left|g_s\right|^2 \frac{\cos(\omega_s t)-1}{\omega_s^2} \\
&= \ln\left[\frac{\left|\left(\frac{k_B T}{\hbar\omega_c}+i\frac{k_B T}{\hbar}t\right)!\right|}{\left(\frac{k_B T}{\hbar\omega_c}\right)!}\right]^{8\eta} .
\end{aligned}$$

Let us define the decoherence function by

$$h(t) \equiv \left\{\frac{\left|\left(\frac{k_B T}{\hbar\omega_c}+i\frac{k_B T}{\hbar}t\right)!\right|^4}{\left(1+\omega_c^2 t^2\right)\left[\left(\frac{k_B T}{\hbar\omega_c}\right)!\right]^4}\right\}^{2\eta} .$$

Thus, in terms of matrix notation, we can write the reduced qubit density matrix as

$$\mathrm{Tr}_B\left[\rho_I(t)\right] = \begin{pmatrix} \left|c_0\right|^2 & c_0 c_1^* h(t) \\ c_0^* c_1 h(t) & \left|c_1\right|^2 \end{pmatrix} .$$

The reader is invited to provide the steps that have been omitted in the above calculations.