A crash introduction to the Wigner function and, then, to the Wigner operator (continuação)

Using Dirac's notation and introducing kets and bras representing, respectively, vectors in the Hilbert space and in its dual, we can rewrite Eq. (1) as

$$w(z,p,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} ds\ \langle\psi(t)|z-s/2\rangle\langle z+s/2|\psi(t)\rangle \exp\left(-i\frac{ps}{\hbar}\right) = \langle\psi(t)|W(z,p)|\psi(t)\rangle, \tag{1}$$

where we have defined the Wigner operator as

$$W(z,p) \equiv \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} ds\ |z-s/2\rangle\langle z+s/2|\exp\left(-i\frac{ps}{\hbar}\right). \tag{2}$$

But, given that

$$|z\rangle = \left(\int_{-\infty}^{\infty} dp\ |p\rangle\langle p|\right)|z\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dp\ |p\rangle \exp(-izp/\hbar)$$

and, of course,

$$\langle z| = \langle z|\left(\int_{-\infty}^{\infty} dp\ |p\rangle\langle p|\right) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dp\ \exp(izp/\hbar)\langle p|,$$

we can also write the Wigner operator as

$$\begin{aligned}
W(z,p) &= \frac{1}{(2\pi\hbar)^2}\int_{-\infty}^{+\infty} ds \int_{-\infty}^{\infty} dp_1\ |p_1\rangle \exp[-i(z-s/2)p_1/\hbar] \\
&\quad\times \int_{-\infty}^{\infty} dp_2\ \exp[i(z+s/2)p_2/\hbar]\langle p_2|\exp\left(-i\frac{ps}{\hbar}\right) \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp_1\ |p_1\rangle \exp(-izp_1/\hbar)\int_{-\infty}^{\infty} dp_2\ \exp(izp_2/\hbar) \\
&\quad\times \langle p_2|\left[\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} ds\ \exp\left[i\frac{s}{\hbar}(p_1+p_2-2p)\right]\right] \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp_1\ |p_1\rangle \exp(-izp_1/\hbar)\exp[iz(2p-p_1)/\hbar]\langle 2p-p_1| \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp'\ |p-p'/2\rangle \exp[-iz(p-p'/2)/\hbar]\exp[iz(p+p'/2)/\hbar]\langle p+p'/2|,
\end{aligned}$$

namely,

$$W(z,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp'\ |p-p'/2\rangle\langle p+p'/2|\exp(izp'/\hbar). \tag{3}$$

A useful point we should notice now is that the Wigner function satisfies an equation of motion that can be derived using Schrödinger's equation. However, to avoid too many calculations, it is now convenient to introduce the operator notation and rewrite the Schrödinger equation in a basis-free representation. To do so, let us introduce the Hamiltonian operator as

$$H = \frac{P^2}{2m} + V(Z), \tag{4}$$

where P is the momentum operator and Z is the position operator. In this operator formalism, the Schrödinger equation becomes

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle. \tag{5}$$

Now we can take the partial time derivative of Eq. (1) to get:

$$\begin{aligned} \frac{\partial}{\partial t} w(z,p,t) &= \left[\frac{\partial}{\partial t} \langle\psi(t)|\right] W(z,p) |\psi(t)\rangle + \langle\psi(t)| W(z,p) \left[\frac{\partial}{\partial t} |\psi(t)\rangle\right] \\ &= -\frac{1}{i\hbar} \langle\psi(t)| H W(z,p) |\psi(t)\rangle + \frac{1}{i\hbar} \langle\psi(t)| W(z,p) H |\psi(t)\rangle \\ &= \frac{1}{i\hbar} \langle\psi(t)| [W(z,p), H] |\psi(t)\rangle, \end{aligned} \tag{6}$$

where we have used the fact that the Hermitian conjugate of Eq. (5) gives

$$-i\hbar \frac{d}{dt} \langle\psi(t)| = \langle\psi(t)| H.$$

To get the equation of motion of the Wigner function we just need to calculate the commutator $[W(z,p), H]$. From Eq. (4) we see that we can split this commutator into two terms:

$$[W(z,p), H] = \left[W(z,p), \frac{P^2}{2m}\right] + [W(z,p), V(Z)]. \tag{7}$$

To calculate the first of these terms, we use Eq. (3):

$$\begin{aligned} \left[W(z,p), \frac{P^2}{2m}\right] &= \frac{1}{2m}\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp' \; [|p-p'/2\rangle\langle p+p'/2|, P^2] \exp(izp'/\hbar) \\ &= \frac{1}{2m}\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp' \; [|p-p'/2\rangle\langle p+p'/2| P^2 - P^2 |p-p'/2\rangle\langle p+p'/2|] \exp(izp'/\hbar) \\ &= \frac{1}{2m}\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp' \; \left[(p+p'/2)^2 - (p-p'/2)^2\right] |p-p'/2\rangle\langle p+p'/2| \exp(izp'/\hbar), \end{aligned}$$

that is,

$$\begin{aligned} \left[W(z,p), \frac{P^2}{2m}\right] &= \frac{p}{m}\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp' \, p' \, |p-p'/2\rangle\langle p+p'/2| \exp(izp'/\hbar) \\ &= -\frac{i\hbar p}{m}\frac{\partial}{\partial z}\left[\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp' \; |p-p'/2\rangle\langle p+p'/2| \exp(izp'/\hbar)\right], \end{aligned}$$

so that

$$\left[W(z,p), \frac{P^2}{2m}\right] = -i\hbar \frac{p}{m}\frac{\partial}{\partial z} W(z,p). \tag{8}$$

Now we calculate the second of the commutators in Eq. (7), but we use the definition of Eq. (2):

$$\begin{aligned}
\left[W\left(z,p\right),V\left(Z\right)\right] &= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} ds\ \left[\left|z-s/2\right\rangle\left\langle z+s/2\right|,V\left(Z\right)\right]\exp\left(-i\frac{ps}{\hbar}\right) \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} ds\ \left[\left|z-s/2\right\rangle\left\langle z+s/2\right|V\left(Z\right)-V\left(Z\right)\left|z-s/2\right\rangle\left\langle z+s/2\right|\right]\exp\left(-i\frac{ps}{\hbar}\right) \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} ds\ \left[\left|z-s/2\right\rangle\left\langle z+s/2\right|V\left(z+s/2\right)-V\left(z-s/2\right)\left|z-s/2\right\rangle\left\langle z+s/2\right|\right] \\
&\quad \times\exp\left(-i\frac{ps}{\hbar}\right),
\end{aligned}$$

so that,

$$\left[W\left(z,p\right),V\left(Z\right)\right] = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} ds\ \left[V\left(z+s/2\right)-V\left(z-s/2\right)\right]\left|z-s/2\right\rangle\left\langle z+s/2\right|\exp\left(-i\frac{ps}{\hbar}\right). \tag{9}$$

Let us notice that, in Eq. (9), the function $V\left(z\right)$, for any $z\in\mathbb{R}$, could – as it will, in the formalism we are going to use shortly – be any operator acting on the Hilbert spaces of the other degrees of freedom other than translational, i.e., position and momentum. For example, this potential could be an operator acting on the internal states of the atom and/or even on the photon states. Whatever kind of entity $V\left(z\right)$ is, however, we will assume that it has a Taylor expansion:

$$V\left(z+\eta\right) = \sum_{n=0}^{\infty}\frac{\eta^{n}}{n!}V^{(n)}\left(z\right), \tag{10}$$

where $V^{(n)}\left(z\right)$ is the nth derivative of $V\left(z\right)$ for $n>0$ and $V^{(0)}\left(z\right)\equiv V\left(z\right)$. By substituting Eq. (10) in Eq. (9) we obtain

$$\begin{aligned}
\left[W\left(z,p\right),V\left(Z\right)\right] &= \sum_{n=0}^{\infty}\frac{\left[1-(-1)^{n}\right]}{n!}V^{(n)}\left(z\right)\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} ds\ \left|z-s/2\right\rangle\left\langle z+s/2\right|\left(\frac{s}{2}\right)^{n}\exp\left(-i\frac{ps}{\hbar}\right) \\
&= \sum_{n=0}^{\infty}\frac{V^{(2n+1)}\left(z\right)}{2^{2n}\left(2n+1\right)!}\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} ds\ \left|z-s/2\right\rangle\left\langle z+s/2\right|s^{2n+1}\exp\left(-i\frac{ps}{\hbar}\right) \\
&= \sum_{n=0}^{\infty}\frac{V^{(2n+1)}\left(z\right)}{2^{2n}\left(2n+1\right)!}\left(i\hbar\right)^{2n+1}\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} ds\ \left|z-s/2\right\rangle\left\langle z+s/2\right|\frac{\partial^{2n+1}}{\partial p^{2n+1}}\exp\left(-i\frac{ps}{\hbar}\right).
\end{aligned}$$

Hence, using Eq. (2), this gives

$$\left[W\left(z,p\right),V\left(Z\right)\right] = i\hbar\sum_{n=0}^{\infty}\frac{V^{(2n+1)}\left(z\right)}{2^{2n}\left(2n+1\right)!}\left(i\hbar\right)^{2n}\frac{\partial^{2n+1}}{\partial p^{2n+1}}W\left(z,p\right). \tag{11}$$

Now we see that Eqs. (4), (8), and (11) result in the following:

$$\left[W\left(z,p\right),H\right] = -i\hbar\frac{p}{m}\frac{\partial}{\partial z}W\left(z,p\right)+i\hbar\sum_{n=0}^{\infty}\frac{V^{(2n+1)}\left(z\right)}{\left(2n+1\right)!}\left(\frac{i\hbar}{2}\right)^{2n}\frac{\partial^{2n+1}}{\partial p^{2n+1}}W\left(z,p\right). \tag{12}$$

Finally, substitution of Eq. (12) into Eq. (6) gives the equation of motion for the Wigner function:

$$\begin{aligned}
\frac{\partial}{\partial t}w\left(z,p,t\right) &= \frac{1}{i\hbar}\left\langle\psi\left(t\right)\right|\left[W\left(z,p\right),H\right]\left|\psi\left(t\right)\right\rangle \\
&= -\frac{p}{m}\frac{\partial}{\partial z}\left\langle\psi\left(t\right)\right|W\left(z,p\right)\left|\psi\left(t\right)\right\rangle+\sum_{n=0}^{\infty}\frac{V^{(2n+1)}\left(z\right)}{\left(2n+1\right)!}\left(\frac{i\hbar}{2}\right)^{2n}\frac{\partial^{2n+1}}{\partial p^{2n+1}}\left\langle\psi\left(t\right)\right|W\left(z,p\right)\left|\psi\left(t\right)\right\rangle,
\end{aligned}$$

that is,

$$\frac{\partial}{\partial t}w\left(z,p,t\right)+\frac{p}{m}\frac{\partial}{\partial z}w\left(z,p,t\right) \quad = \quad \sum_{n=0}^{\infty}\frac{V^{(2n+1)}\left(z\right)}{(2n+1)!}\left(\frac{i\hbar}{2}\right)^{2n}\frac{\partial^{2n+1}}{\partial p^{2n+1}}w\left(z,p,t\right), \tag{13}$$

where we have used the definition of Eq. (1).

We have, thus, found an equation that has all even powers of $\hbar$ on its right-hand side, showing, explicitly, that it describes a fully quantum translational motion. Let us notice that these powers of $\hbar$ refer only to the quantum character due to the translational motion, and not to eventual factors of $\hbar$ coming from the other quantum degrees of freedom, as we shall see. We say that our approach is semi-classical in the extent that we neglect the quantum dynamics of the translational motion of the atomic center of mass. In the present formalism this approximation simply amounts to ignoring all the factors explicitly proportional to $\hbar$ in Eq. (13), illustrating the usefulness of the Wigner-operator approach. Hence, Eq. (13) can now be approximated as:

$$\frac{\partial}{\partial t}w\left(z,p,t\right)+\frac{p}{m}\frac{\partial}{\partial z}w\left(z,p,t\right) \quad \approx \quad \frac{\partial V\left(z\right)}{\partial z}\frac{\partial}{\partial p}w\left(z,p,t\right). \tag{14}$$

In the context in which we have started this discussion, in which we do not have additional degrees of freedom, Eq. (14) is a c-number equation and $w\left(z,p,t\right)$ is simply a real function. Equation (14) shows that the Wigner function is equivalent to our classical probability distribution satisfying Liouville's dynamics, as given by

$$\frac{\partial}{\partial t}\rho_{cl}\left(z,p,t\right)+\frac{p}{m}\frac{\partial}{\partial z}\rho_{cl}\left(z,p,t\right) \quad = \quad \frac{\partial V\left(z\right)}{\partial z}\frac{\partial}{\partial p}\rho_{cl}\left(z,p,t\right). \tag{15}$$

However, we can follow a very similar procedure when $V\left(z\right)$ is an operator function acting on degrees of freedom other than the translational ones. To make this more precise, for such a more general potential operator, instead of using Eq. (12), when we use the semi-classical approach mentioned above, in which only the translation is treated as classical, we will adopt the following approximation:

$$\left[W\left(z,p\right),H\right] \quad \approx \quad -i\hbar\frac{p}{m}\frac{\partial}{\partial z}W\left(z,p\right)+i\hbar\frac{\partial V\left(z\right)}{\partial z}\frac{\partial}{\partial p}W\left(z,p\right). \tag{16}$$