Equações Ópticas de Bloch

Agora vamos usar a equação acima no caso de nosso átomo de dois níveis com um laser, incluindo emissão espontânea. Na aula 11 tínhamos a hamiltoniana de interação entre a radiação e a matéria:

$$\begin{aligned} H_I^{RM}(t) &= -\hbar\frac{\Omega_L}{2}\left\{\exp\left[i\left(\omega_L-\omega_{at}\right)t\right]|0\rangle\langle 1|+\exp\left[-i\left(\omega_L-\omega_{at}\right)t\right]|1\rangle\langle 0|\right\} \\ &\quad -\hbar\sum_{\mathbf{k},\lambda}\left\{ig_{\mathbf{k},\lambda}|1\rangle\langle 0|a_{\mathbf{k},\lambda}\exp\left[-i\left(\omega_{\mathbf{k},\lambda}-\omega_{at}\right)t\right]\right. \\ &\quad \left.-ig^*_{\mathbf{k},\lambda}|0\rangle\langle 1|a^\dagger_{\mathbf{k},\lambda}\exp\left[i\left(\omega_{\mathbf{k},\lambda}-\omega_{at}\right)t\right]\right\}. \end{aligned} \tag{1}$$

Vamos considerar dois termos dessa hamiltoniana:

$$H_L(t) \equiv -\hbar\frac{\Omega_L}{2}\left\{\exp\left[i\left(\omega_L-\omega_{at}\right)t\right]|0\rangle\langle 1|+\exp\left[-i\left(\omega_L-\omega_{at}\right)t\right]|1\rangle\langle 0|\right\} \tag{2}$$

e

$$\begin{aligned} H_I(t) &\equiv -\hbar\sum_{\mathbf{k},\lambda}\left\{ig_{\mathbf{k},\lambda}|1\rangle\langle 0|a_{\mathbf{k},\lambda}\exp\left[-i\left(\omega_{\mathbf{k},\lambda}-\omega_{at}\right)t\right]\right. \\ &\quad \left.-ig^*_{\mathbf{k},\lambda}|0\rangle\langle 1|a^\dagger_{\mathbf{k},\lambda}\exp\left[i\left(\omega_{\mathbf{k},\lambda}-\omega_{at}\right)t\right]\right\} \end{aligned} \tag{3}$$

e agora

$$H_I^{RM}(t) = H_L(t)+H_I(t).$$

Vamos recordar a equação de Liouville:

$$i\hbar\frac{d}{dt}\rho_I(t) = \left[H_I^{RM}(t),\rho_I(t)\right].$$

No caso com o laser acima, podemos, ao invés, escrever

$$i\hbar\frac{d}{dt}\rho_I(t) = \left[H_L(t),\rho_I(t)\right]+\left[H_I(t),\rho_I(t)\right]. \tag{4}$$

A interação com o laser é forte, ao contrário da interação com o vácuo. Para poder usar a equação mestra que deduzimos nas aulas anteriores no caso da Eq. (4), vamos fazer uma nova transformação unitária. Seja o operador $U_L(t)$ dado por

$$i\hbar\frac{d}{dt}U_L(t) = H_L(t)\,U_L(t),$$

com

$$U_L(0) = I.$$

Então, definimos

$$\rho_{IL}(t) = U_L^\dagger(t)\,\rho_I(t)\,U_L(t).$$

Portanto,

$$\begin{aligned} i\hbar\frac{d}{dt}\rho_{IL}(t) &= -U_L^\dagger(t)\,H_L(t)\,\rho_I(t)\,U_L(t)+U_L^\dagger(t)\,\rho_I(t)\,H_L(t)\,U_L(t)+U_L^\dagger(t)\left[i\hbar\frac{d}{dt}\rho_I(t)\right]U_L(t)\\ &= U_L^\dagger(t)\left\{-\left[H_L(t),\rho_I(t)\right]+i\hbar\frac{d}{dt}\rho_I(t)\right\}U_L(t)\\ &= U_L^\dagger(t)\left\{-\left[H_L(t),\rho_I(t)\right]+\left[H_L(t),\rho_I(t)\right]+\left[H_I(t),\rho_I(t)\right]\right\}U_L(t)\\ &= U_L^\dagger(t)\left[H_I(t),\rho_I(t)\right]U_L(t). \end{aligned}$$

Mas,

$$\begin{aligned} U_L^\dagger(t)\left[H_I(t),\rho_I(t)\right]U_L(t) &= \left[U_L^\dagger(t)\,H_I(t)\,U_L(t),U_L^\dagger(t)\,\rho_I(t)\,U_L(t)\right]\\ &= \left[U_L^\dagger(t)\,H_I(t)\,U_L(t),\rho_{IL}(t)\right]. \end{aligned}$$

Definamos, então,

$$H_{IL}(t) = U_L^\dagger(t)\,H_I(t)\,U_L(t).$$

A equação mestra nessa nova representação é dada por

$$i\hbar\frac{d}{dt}\rho_{IL}(t) = \left[H_{IL}(t),\rho_{IL}(t)\right].$$

Como $H_I(t)$ é proporcional a $|g_{\mathbf{k},\lambda}|$, segue que $H_{IL}(t)$ também é e, assim, usamos a equação mestra que deduzimos anteriormente, obtendo:

$$\frac{d}{dt}\rho_{ILS}(t) = -\frac{1}{\hbar^2}\mathrm{Tr}_B\left\{\int_0^t dt'\,\left[H_{IL}(t),\left[H_{IL}(t'),\rho_B(0)\,\rho_{ILS}(t)\right]\right]\right\},$$

onde agora o operador densidade reduzido do átomo é dado por

$$\rho_{ILS}(t) = \mathrm{Tr}_B\left[\rho_{IL}(t)\right].$$

Consideremos o integrando acima:

$$\left[H_{IL}(t),\left[H_{IL}(t'),\rho_B(0)\,\rho_{ILS}(t)\right]\right] = \left[U_L^\dagger(t)\,H_I(t)\,U_L(t),\left[U_L^\dagger(t')\,H_I(t')\,U_L(t'),\rho_B(0)\,\mathrm{Tr}_B\left[U_L^\dagger(t)\,\rho_I(t)\,U_L(t)\right]\right]\right].$$

Quando o laser está bem próximo à ressonância atômica, $\omega_L-\omega_{at}$ é uma frequência de magnitude pequena. Além disso, as frequências de Rabi que usamos no laboratório para átomos alcalinos, como o sódio e o rubídio, por exemplo, são da ordem de 10 MHz. Já os fótons virtuais podem ter frequências que tais que $(\omega_{\mathbf{k},\lambda}-\omega_{at})$ pode ser da ordem de ω_{at} e isso corresponde a 10^{15} Hz. Sendo assim, o integrando acima vai anular a integral em tempos curtos comparados com $1/\Omega_L$ e, dessa forma, dentro do integrando e somente dentro do integrando, podemos aproximar:

$$U_L(t') \approx U_L(t).$$

Com isso, vemos que o comutador acima se reduz a

$$\left[H_{IL}(t),\left[H_{IL}(t'),\rho_B(0)\,\rho_{ILS}(t)\right]\right] \approx \left[U_L^\dagger(t)\,H_I(t)\,U_L(t),\left[U_L^\dagger(t)\,H_I(t')\,U_L(t),\rho_B(0)\,\mathrm{Tr}_B\left[U_L^\dagger(t)\,\rho_I(t)\,U_L(t)\right]\right]\right]$$

$$
\begin{aligned}
&= \left[U_L^\dagger(t)\, H_I(t)\, U_L(t)\,, \left[U_L^\dagger(t)\, H_I(t')\, U_L(t)\,, \rho_B(0)\, U_L^\dagger(t)\, \mathrm{Tr}_B\left[\rho_I(t)\right] U_L(t)\right]\right] \\
&= \left[U_L^\dagger(t)\, H_I(t)\, U_L(t)\,, U_L^\dagger(t)\left[H_I(t')\,, \rho_B(0)\, \mathrm{Tr}_B\left[\rho_I(t)\right]\right] U_L(t)\right] \\
&= \left[U_L^\dagger(t)\, H_I(t)\, U_L(t)\,, U_L^\dagger(t)\left[H_I(t')\,, \rho_B(0)\, \rho_{IS}(t)\right] U_L(t)\right] \\
&= U_L^\dagger(t)\left[H_I(t)\,, \left[H_I(t')\,, \rho_B(0)\, \rho_{IS}(t)\right]\right] U_L(t)\,.
\end{aligned}
$$

Logo, a equação mestra acima simplifica e agora pode ser escrita como

$$
\begin{aligned}
\frac{d}{dt}\rho_{ILS}(t) &\approx -\frac{1}{\hbar^2}\mathrm{Tr}_B\left\{\int_0^t dt'\, U_L^\dagger(t)\left[H_I(t)\,, \left[H_I(t')\,, \rho_B(0)\,\rho_{IS}(t)\right]\right] U_L(t)\right\} \\
&= -\frac{1}{\hbar^2} U_L^\dagger(t)\, \mathrm{Tr}_B\left\{\int_0^t dt'\ \left[H_I(t)\,, \left[H_I(t')\,, \rho_B(0)\,\rho_{IS}(t)\right]\right]\right\} U_L(t)\,.
\end{aligned}
$$

Mas,

$$
\begin{aligned}
\frac{d}{dt}\rho_{ILS}(t) &= \frac{d}{dt}\mathrm{Tr}_B\left[\rho_{IL}(t)\right] \\
&= \frac{d}{dt}\mathrm{Tr}_B\left[U_L(t)\,\rho_I(t)\, U_L^\dagger(t)\right] \\
&= \mathrm{Tr}_B\left\{\frac{d}{dt}\left[U_L(t)\,\rho_I(t)\, U_L^\dagger(t)\right]\right\}.
\end{aligned}
$$

Como vimos acima,

$$
i\hbar\frac{d}{dt}\rho_{IL}(t) = U_L^\dagger(t)\left\{-\left[H_L(t)\,, \rho_I(t)\right] + i\hbar\frac{d}{dt}\rho_I(t)\right\} U_L(t)
$$

e, assim,

$$
\frac{d}{dt}\left[U_L(t)\,\rho_I(t)\, U_L^\dagger(t)\right] = U_L^\dagger(t)\left\{\frac{i}{\hbar}\left[H_L(t)\,, \rho_I(t)\right] + \frac{d}{dt}\rho_I(t)\right\} U_L(t)\,.
$$

Com isso, a equação acima agora fica

$$
\mathrm{Tr}_B\left\{\frac{d}{dt}\left[U_L(t)\,\rho_I(t)\, U_L^\dagger(t)\right]\right\} \approx -\frac{1}{\hbar^2} U_L^\dagger(t)\, \mathrm{Tr}_B\left\{\int_0^t dt'\ \left[H_I(t)\,, \left[H_I(t')\,, \rho_B(0)\,\rho_{IS}(t)\right]\right]\right\} U_L(t)\,,
$$

que é equivalente a

$$
U_L^\dagger(t)\, \mathrm{Tr}_B\left\{\frac{i}{\hbar}\left[H_L(t)\,, \rho_I(t)\right] + \frac{d}{dt}\rho_I(t)\right\} U_L(t) \approx -\frac{1}{\hbar^2} U_L^\dagger(t)\, \mathrm{Tr}_B\left\{\int_0^t dt'\ \left[H_I(t)\,, \left[H_I(t')\,, \rho_B(0)\,\rho_{IS}(t)\right]\right]\right\} U_L(t)\,.
$$

Cancelando os operadores unitários, obtemos a equação mestra que dá origem às equações ópticas de Bloch:

$$
\frac{d}{dt}\rho_{IS}(t) \approx \frac{1}{i\hbar}\left[H_L(t)\,, \rho_{IS}(t)\right] - \frac{1}{\hbar^2}\mathrm{Tr}_B\left\{\int_0^t dt'\ \left[H_I(t)\,, \left[H_I(t')\,, \rho_B(0)\,\rho_{IS}(t)\right]\right]\right\}, \tag{5}
$$

onde notamos que

$$
\begin{aligned}
\mathrm{Tr}_B\left\{\left[H_L(t)\,, \rho_I(t)\right]\right\} &= \left[H_L(t)\,, \mathrm{Tr}_B\left[\rho_I(t)\right]\right] \\
&= \left[H_L(t)\,, \rho_{IS}(t)\right].
\end{aligned}
$$

A dificuldade de deduzir as equações ópticas de Bloch está na integral, que dá origem ao chamado termo lindbladiano. Notemos que teremos que calcular o traço e fazer a integral, independentemente da ordem. Vamos recordar a aula 11 e escrever:

$$A(t) = -\sum_{\mathbf{k},\lambda} i\hbar g_{\mathbf{k},\lambda} \exp\left[-i(\omega_k - \omega_{at})t\right] a_{\mathbf{k},\lambda},$$

$$A^{\dagger}(t) = \sum_{\mathbf{k},\lambda} i\hbar g_{\mathbf{k},\lambda} \exp\left[i(\omega_k - \omega_{at})t\right] a^{\dagger}_{\mathbf{k},\lambda}$$

e

$$H_I(t) = |1\rangle\langle 0| A(t) + |0\rangle\langle 1| A^{\dagger}(t),$$

já usando $g_{\mathbf{k},\lambda} \in \mathbb{R}$. Assim, escrevemos agora:

$$\begin{aligned}
\left[H_I(t), \left[H_I(t'), \rho_B(0)\rho_{IS}(t)\right]\right] &= \left[|1\rangle\langle 0| A(t) + |0\rangle\langle 1| A^{\dagger}(t), \left[|1\rangle\langle 0| A(t') + |0\rangle\langle 1| A^{\dagger}(t'), \rho_B(0)\rho_{IS}(t)\right]\right] \\
&= \left[|1\rangle\langle 0| A(t), \left[|1\rangle\langle 0| A(t'), \rho_B(0)\rho_{IS}(t)\right]\right] \\
&\quad + \left[|0\rangle\langle 1| A^{\dagger}(t), \left[|1\rangle\langle 0| A(t'), \rho_B(0)\rho_{IS}(t)\right]\right] \\
&\quad + \left[|1\rangle\langle 0| A(t), \left[|0\rangle\langle 1| A^{\dagger}(t'), \rho_B(0)\rho_{IS}(t)\right]\right] \\
&\quad + \left[|0\rangle\langle 1| A^{\dagger}(t), \left[|0\rangle\langle 1| A^{\dagger}(t'), \rho_B(0)\rho_{IS}(t)\right]\right].
\end{aligned}$$

Quando tomarmos o traço sobre os fótons, só os termos que preservam o número de fótons é que contribuirão. Com isso, podemos escrever agora que

$$\begin{aligned}
\mathrm{Tr}_B\left\{\left[H_I(t), \left[H_I(t'), \rho_B(0)\rho_{IS}(t)\right]\right]\right\} &= \mathrm{Tr}_B\left\{\left[|0\rangle\langle 1| A^{\dagger}(t), \left[|1\rangle\langle 0| A(t'), \rho_B(0)\rho_{IS}(t)\right]\right]\right\} \\
&\quad + \mathrm{Tr}_B\left\{\left[|1\rangle\langle 0| A(t), \left[|0\rangle\langle 1| A^{\dagger}(t'), \rho_B(0)\rho_{IS}(t)\right]\right]\right\}.
\end{aligned}$$

Só precisamos olhar o que dá um dos termos, já que o segundo é seu conjugado hermitiano. Sendo assim, escrevemos:

$$\begin{aligned}
\mathrm{Tr}_B\left\{\left[|0\rangle\langle 1| A^{\dagger}(t), \left[|1\rangle\langle 0| A(t'), \rho_B(0)\rho_{IS}(t)\right]\right]\right\} &= \mathrm{Tr}_B\left\{|0\rangle\langle 1| A^{\dagger}(t)\left[|1\rangle\langle 0| A(t'), \rho_B(0)\rho_{IS}(t)\right]\right\} \\
&\quad - \mathrm{Tr}_B\left\{\left[|1\rangle\langle 0| A(t'), \rho_B(0)\rho_{IS}(t)\right]|0\rangle\langle 1| A^{\dagger}(t)\right\} \\
&= \mathrm{Tr}_B\left\{|0\rangle\langle 1| A^{\dagger}(t)|1\rangle\langle 0| A(t')\rho_B(0)\rho_{IS}(t)\right\} \\
&\quad - \mathrm{Tr}_B\left\{|0\rangle\langle 1| A^{\dagger}(t)\rho_B(0)\rho_{IS}(t)|1\rangle\langle 0| A(t')\right\} \\
&\quad - \mathrm{Tr}_B\left\{|1\rangle\langle 0| A(t')\rho_B(0)\rho_{IS}(t)|0\rangle\langle 1| A^{\dagger}(t)\right\} \\
&\quad + \mathrm{Tr}_B\left\{\rho_B(0)\rho_{IS}(t)|1\rangle\langle 0| A(t')|0\rangle\langle 1| A^{\dagger}(t)\right\} \\
&= \mathrm{Tr}_B\left\{A^{\dagger}(t)A(t')\rho_B(0)\right\}|0\rangle\langle 0|\rho_{IS}(t) \\
&\quad - \mathrm{Tr}_B\left\{A^{\dagger}(t)\rho_B(0)A(t')\right\}|0\rangle\langle 1|\rho_{IS}(t)|1\rangle\langle 0| \\
&\quad - \mathrm{Tr}_B\left\{A(t')\rho_B(0)A^{\dagger}(t)\right\}|1\rangle\langle 0|\rho_{IS}(t)|0\rangle\langle 1| \\
&\quad + \mathrm{Tr}_B\left\{\rho_B(0)A(t')A^{\dagger}(t)\right\}\rho_{IS}(t)|1\rangle\langle 1|.
\end{aligned}$$

Só para conferir, calculemos também:

$$\mathrm{Tr}_B\left\{\left[|1\rangle\langle 0| A(t), \left[|0\rangle\langle 1| A^{\dagger}(t'), \rho_B(0)\rho_{IS}(t)\right]\right]\right\} = \mathrm{Tr}_B\left\{|1\rangle\langle 0| A(t)\left[|0\rangle\langle 1| A^{\dagger}(t'), \rho_B(0)\rho_{IS}(t)\right]\right\}$$

$$
\begin{aligned}
&-\mathrm{Tr}_B\left\{\left[|0\rangle\langle 1| A^{\dagger}(t'), \rho_B(0)\rho_{IS}(t)\right]|1\rangle\langle 0| A(t)\right\} \\
=\;& \mathrm{Tr}_B\left\{|1\rangle\langle 0| A(t)|0\rangle\langle 1| A^{\dagger}(t')\rho_B(0)\rho_{IS}(t)\right\} \\
&-\mathrm{Tr}_B\left\{|1\rangle\langle 0| A(t)\rho_B(0)\rho_{IS}(t)|0\rangle\langle 1| A^{\dagger}(t')\right\} \\
&-\mathrm{Tr}_B\left\{|0\rangle\langle 1| A^{\dagger}(t')\rho_B(0)\rho_{IS}(t)|1\rangle\langle 0| A(t)\right\} \\
&+\mathrm{Tr}_B\left\{\rho_B(0)\rho_{IS}(t)|0\rangle\langle 1| A^{\dagger}(t')|1\rangle\langle 0| A(t)\right\} \\
=\;& \mathrm{Tr}_B\left\{A(t)A^{\dagger}(t')\rho_B(0)\right\}|1\rangle\langle 1|\rho_{IS}(t) \\
&-\mathrm{Tr}_B\left\{A(t)\rho_B(0)A^{\dagger}(t')\right\}|1\rangle\langle 0|\rho_{IS}(t)|0\rangle\langle 1| \\
&-\mathrm{Tr}_B\left\{A^{\dagger}(t')\rho_B(0)A(t)\right\}|0\rangle\langle 1|\rho_{IS}(t)|1\rangle\langle 0| \\
&+\mathrm{Tr}_B\left\{\rho_B(0)A^{\dagger}(t')A(t)\right\}\rho_{IS}(t)|0\rangle\langle 0|,
\end{aligned}
$$

como esperado.

Só precisamos calcular um desses traços, pois do resultado obteremos os demais. Para vermos isso, notemos que

$$
\begin{aligned}
\left[A(t), A^{\dagger}(t')\right] &= \left[-\sum_{\mathbf{k},\lambda} i\hbar g_{\mathbf{k},\lambda}\exp\left[-i(\omega_k-\omega_{at})t\right]a_{\mathbf{k},\lambda}, \sum_{\mathbf{k}',\lambda'} i\hbar g_{\mathbf{k}',\lambda'}\exp\left[i(\omega_{k'}-\omega_{at})t'\right]a^{\dagger}_{\mathbf{k}',\lambda'}\right] \\
&= \hbar^2\sum_{\mathbf{k},\lambda} g_{\mathbf{k},\lambda}\sum_{\mathbf{k}',\lambda'} g_{\mathbf{k}',\lambda'}\exp\left[i(\omega_{k'}-\omega_{at})t' - i(\omega_k-\omega_{at})t\right]\left[a_{\mathbf{k},\lambda}, a^{\dagger}_{\mathbf{k}',\lambda'}\right] \\
&= \hbar^2\sum_{\mathbf{k},\lambda}\left|g_{\mathbf{k},\lambda}\right|^2\exp\left[-i(\omega_k-\omega_{at})(t-t')\right].
\end{aligned}
$$

Logo,

$$
A(t)A^{\dagger}(t') = A^{\dagger}(t')A(t) + \hbar^2\sum_{\mathbf{k},\lambda}\left|g_{\mathbf{k},\lambda}\right|^2\exp\left[-i(\omega_k-\omega_{at})(t-t')\right]
$$

e, portanto,

$$
\begin{aligned}
\mathrm{Tr}_B\left[\rho_B(0)A(t)A^{\dagger}(t')\right] &= \mathrm{Tr}_B\left[\rho_B(0)A^{\dagger}(t')A(t)\right] + \mathrm{Tr}_B\left[\rho_B(0)\right]\hbar^2\sum_{\mathbf{k},\lambda}\left|g_{\mathbf{k},\lambda}\right|^2\exp\left[-i(\omega_k-\omega_{at})(t-t')\right] \\
&= \mathrm{Tr}_B\left[\rho_B(0)A^{\dagger}(t')A(t)\right] + \hbar^2\sum_{\mathbf{k},\lambda}\left|g_{\mathbf{k},\lambda}\right|^2\exp\left[-i(\omega_k-\omega_{at})(t-t')\right].
\end{aligned}
$$

Os demais são apenas variações desses dois traços com a troca de entre t e t'. Também notemos que

$$
\begin{aligned}
\mathrm{Tr}_B\left[\rho_B(0)A(t)A^{\dagger}(t')\right] &= \hbar^2\mathrm{Tr}_B\left[\rho_B(0)\sum_{\mathbf{k},\lambda} g_{\mathbf{k},\lambda}\exp\left[-i(\omega_k-\omega_{at})t\right]\sum_{\mathbf{k}',\lambda'} g_{\mathbf{k}',\lambda'}\exp\left[i(\omega_{k'}-\omega_{at})t'\right]a_{\mathbf{k},\lambda}a^{\dagger}_{\mathbf{k}',\lambda'}\right] \\
&= \hbar^2\sum_{\mathbf{k},\lambda}\left|g_{\mathbf{k},\lambda}\right|^2\exp\left[-i(\omega_k-\omega_{at})(t-t')\right]\mathrm{Tr}_B\left[\rho_B(0)a_{\mathbf{k},\lambda}a^{\dagger}_{\mathbf{k},\lambda}\right].
\end{aligned}
$$

Com isso, vamos definir a função

$$
\begin{aligned}
\mathcal{G}(\tau) &= \mathrm{Tr}_B\left[\rho_B(0)A(\tau)A^{\dagger}(0)\right] \\
&= \hbar^2\sum_{\mathbf{k},\lambda}\left|g_{\mathbf{k},\lambda}\right|^2\exp\left[-i(\omega_k-\omega_{at})\tau\right]\mathrm{Tr}_B\left[\rho_B(0)a_{\mathbf{k},\lambda}a^{\dagger}_{\mathbf{k},\lambda}\right].
\end{aligned}
$$

Seja

$$\mathcal{J}(\tau) = \hbar^2 \sum_{\mathbf{k},\lambda} |g_{\mathbf{k},\lambda}|^2 \exp\left[-i(\omega_k - \omega_{at})\tau\right].$$

Na aula 14 vimos que

$$\rho_B(0) = \frac{1}{Z} \exp\left(-\frac{H_B}{k_B T}\right),$$

onde

$$H_B = \sum_{\mathbf{k},\lambda} \hbar\omega_k a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda}$$

e

$$Z = \mathrm{Tr}_B\left[\exp\left(-\frac{H_B}{k_B T}\right)\right].$$

Portanto,

$$\begin{aligned}\mathrm{Tr}_B\left[\rho_B(0)\, a_{\mathbf{k},\lambda} a^\dagger_{\mathbf{k},\lambda}\right] &= 1 + \mathrm{Tr}_B\left[\rho_B(0)\, a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda}\right] \\ &= 1 + \langle n_{\mathbf{k},\lambda}\rangle,\end{aligned}$$

onde $\langle n_{\mathbf{k},\lambda}\rangle$ é o valor esperado do operador número do modo $(\mathbf{k},\lambda)$ do vácuo eletromagnético,

$$n_{\mathbf{k},\lambda} = a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda},$$

no estado térmico $\rho_B(0)$. Em outras palavras, $\langle n_{\mathbf{k},\lambda}\rangle$ é a ocupação esperada do modo $(\mathbf{k},\lambda)$ à temperatura T. Assim,

$$\mathcal{G}(\tau) = \hbar^2 \sum_{\mathbf{k},\lambda} |g_{\mathbf{k},\lambda}|^2 \exp\left[-i(\omega_k - \omega_{at})\tau\right] \langle n_{\mathbf{k},\lambda}\rangle + \mathcal{J}(\tau).$$

Agora vem a dificuldade de fazer a integral no tempo. Vamos considerar uma dessas integrais, por exemplo:

$$\mathcal{L} = \int_0^t dt' \, \exp\left[i(\omega - \omega_{at})(t - t')\right],$$

tomando, para simplificar a notação, $\omega_k = \omega$, pois logo tomaremos o limite do contínuo e a soma sobre $\mathbf{k}$ se torna uma integral. Agora temos que investigar como lidar com o problema de que há um ponto na integração sobre ω em que $\omega = \omega_{at}$. Neste ponto, a integral no tempo dá t, mas só neste valor da integral sobre ω. Quando $\omega \neq \omega_{at}$, aí temos

$$\begin{aligned}\int_0^t dt' \, \exp\left[i(\omega - \omega_{at})(t - t')\right] &= \exp\left[i(\omega - \omega_{at})t\right] \int_0^t dt' \, \exp\left[-i(\omega - \omega_{at})t'\right] \\ &= i\frac{1 - \exp\left[i(\omega - \omega_{at})t\right]}{\omega - \omega_{at}},\end{aligned}$$

que oscila rapidamente para tempos longos comparados com $1/\omega_{at}$, que é da ordem de 10^{-15} s^{-1}.

Uma estratégia é usarmos

$$
\begin{aligned}
\int_0^t dt' \, \exp\left[i\left(\omega-\omega_{at}\right)\left(t-t'\right)\right] &\approx \lim_{t\to+\infty}\int_0^t dt' \, \exp\left[i\left(\omega-\omega_{at}\right)\left(t-t'\right)\right] \\
&= \lim_{\eta\to0^+}\lim_{t\to+\infty}\int_0^t dt' \, \exp\left[i\left(\omega-\omega_{at}+i\eta\right)\left(t-t'\right)\right] \\
&= \lim_{\eta\to0^+}\lim_{t\to+\infty}\Big\{\exp\left[i\left(\omega-\omega_{at}+i\eta\right)t\right] \\
&\quad \times\int_0^t dt' \, \exp\left[-i\left(\omega-\omega_{at}+i\eta\right)t'\right]\Big\} \\
&= \lim_{\eta\to0^+}\lim_{t\to+\infty}\Bigg\{\exp\left[i\left(\omega-\omega_{at}+i\eta\right)t\right] \\
&\quad \times\frac{\exp\left[-i\left(\omega-\omega_{at}+i\eta\right)t\right]-1}{-i\left(\omega-\omega_{at}+i\eta\right)}\Bigg\} \\
&= \lim_{\eta\to0^+}\lim_{t\to+\infty}\left\{\frac{1-\exp\left[i\left(\omega-\omega_{at}+i\eta\right)t\right]}{-i\left(\omega-\omega_{at}+i\eta\right)}\right\} \\
&= \lim_{\eta\to0^+}\frac{i}{\left(\omega-\omega_{at}+i\eta\right)} \\
&= -i\lim_{\eta\to0^+}\frac{1}{\omega_{at}-\omega-i\eta} \\
&= -i\left[-\mathscr{P}\frac{1}{\omega-\omega_{at}}+i\pi\delta\left(\omega-\omega_{at}\right)\right] \\
&= \mathscr{P}\frac{i}{\omega-\omega_{at}}+\pi\delta\left(\omega-\omega_{at}\right).
\end{aligned}
$$

Assim,

$$
\begin{aligned}
\mathcal{L} &= \int_0^t dt' \, \exp\left[i\left(\omega-\omega_{at}\right)\left(t-t'\right)\right] \\
&\approx \mathscr{P}\frac{i}{\omega-\omega_{at}}+\pi\delta\left(\omega-\omega_{at}\right).
\end{aligned}
$$

Usando esse resultado podemos escrever:

$$
\begin{aligned}
\int_0^t dt' \, \mathcal{J}\left(t-t'\right) &\approx \hbar^2\sum_{\mathbf{k},\lambda}\left|g_{\mathbf{k},\lambda}\right|^2\left[-\mathscr{P}\frac{i}{\omega_k-\omega_{at}}+\pi\delta\left(\omega_k-\omega_{at}\right)\right] \\
&= -i\hbar^2\mathscr{P}\sum_{\mathbf{k},\lambda}\frac{\left|g_{\mathbf{k},\lambda}\right|^2}{\omega_k-\omega_{at}}+\pi\hbar^2\sum_{\mathbf{k},\lambda}\left|g_{\mathbf{k},\lambda}\right|^2\delta\left(\omega_k-\omega_{at}\right).
\end{aligned}
$$

Também,

$$
\int_0^t dt' \, \mathcal{G}\left(t-t'\right) = \hbar^2\sum_{\mathbf{k},\lambda}\left|g_{\mathbf{k},\lambda}\right|^2\int_0^t dt' \, \exp\left[-i\left(\omega_k-\omega_{at}\right)\left(t-t'\right)\right]\left\langle n_{\mathbf{k},\lambda}\right\rangle+\int_0^t dt' \, \mathcal{J}\left(t-t'\right)
$$

$$\begin{aligned}
&\approx \hbar^2 \sum_{\mathbf{k},\lambda} \left|g_{\mathbf{k},\lambda}\right|^2 \left[-\mathscr{P}\frac{i}{\omega_k-\omega_{at}} + \pi\delta\left(\omega_k-\omega_{at}\right)\right]\langle n_{\mathbf{k},\lambda}\rangle \\
&\quad -i\hbar^2\mathscr{P}\sum_{\mathbf{k},\lambda}\frac{\left|g_{\mathbf{k},\lambda}\right|^2}{\omega_k-\omega_{at}} + \pi\hbar^2\sum_{\mathbf{k},\lambda}\left|g_{\mathbf{k},\lambda}\right|^2\delta\left(\omega_k-\omega_{at}\right) \\
&= -i\hbar^2\mathscr{P}\sum_{\mathbf{k},\lambda}\frac{\left|g_{\mathbf{k},\lambda}\right|^2\langle n_{\mathbf{k},\lambda}\rangle}{\omega_k-\omega_{at}} + \pi\hbar^2\sum_{\mathbf{k},\lambda}\left|g_{\mathbf{k},\lambda}\right|^2\langle n_{\mathbf{k},\lambda}\rangle\delta\left(\omega_k-\omega_{at}\right) \\
&\quad -i\hbar^2\mathscr{P}\sum_{\mathbf{k},\lambda}\frac{\left|g_{\mathbf{k},\lambda}\right|^2}{\omega_k-\omega_{at}} + \pi\hbar^2\sum_{\mathbf{k},\lambda}\left|g_{\mathbf{k},\lambda}\right|^2\delta\left(\omega_k-\omega_{at}\right) \\
&= -i\hbar^2\mathscr{P}\sum_{\mathbf{k},\lambda}\frac{\left|g_{\mathbf{k},\lambda}\right|^2\left(\langle n_{\mathbf{k},\lambda}\rangle+1\right)}{\omega_k-\omega_{at}} + \pi\hbar^2\sum_{\mathbf{k},\lambda}\left|g_{\mathbf{k},\lambda}\right|^2\left(\langle n_{\mathbf{k},\lambda}\rangle+1\right)\delta\left(\omega_k-\omega_{at}\right).
\end{aligned}$$

Sejam agora:

$$\Gamma_T = \pi\sum_{\mathbf{k},\lambda}\left|g_{\mathbf{k},\lambda}\right|^2\langle n_{\mathbf{k},\lambda}\rangle\delta\left(\omega_k-\omega_{at}\right),$$

$$\Xi_T = \mathscr{P}\sum_{\mathbf{k},\lambda}\frac{\left|g_{\mathbf{k},\lambda}\right|^2\langle n_{\mathbf{k},\lambda}\rangle}{\omega_k-\omega_{at}},$$

$$\Gamma_0 = \pi\sum_{\mathbf{k},\lambda}\left|g_{\mathbf{k},\lambda}\right|^2\delta\left(\omega_k-\omega_{at}\right)$$

e

$$\Xi_0 = \mathscr{P}\sum_{\mathbf{k},\lambda}\frac{\left|g_{\mathbf{k},\lambda}\right|^2}{\omega_k-\omega_{at}}.$$

Então,

$$\int_0^t dt'\,\mathcal{J}\left(t-t'\right) \approx -i\hbar^2\Xi_0+\hbar^2\Gamma_0$$

e

$$\int_0^t dt'\,\mathcal{G}\left(t-t'\right) \approx -i\hbar^2\Xi_T+\hbar^2\Gamma_T-i\hbar^2\Xi_0+\hbar^2\Gamma_0.$$