Back on tracks (continuação)

Hence,

$$\begin{aligned}
\mathrm{Tr}_B\left\{\left[H_I(t),\rho_I(0)\right]\right\} &= \mathrm{Tr}_B\left\{\left[H_I(t),\rho_S(0)\rho_B(0)\right]\right\} \\
&= \rho_S(0)\,\mathrm{Tr}_B\left\{\left[H_I(t),\rho_B(0)\right]\right\} \\
&\quad +\mathrm{Tr}_B\left\{\left[H_I(t),\rho_S(0)\right]\rho_B(0)\right\}.
\end{aligned}$$

Because of the our choice of $\rho_B(0)$ given above and because $H_I(t)$ is linear in the boson operators, it follows that

$$\mathrm{Tr}_B\left\{\left[H_I(t),\rho_B(0)\right]\right\} = 0.$$

Notice that, since

$$\rho_B(0) = \sum_i p_i\left|E_i\right\rangle\left\langle E_i\right|,$$

it follows, for instance, that

$$\begin{aligned}
\mathrm{Tr}_B\left\{H_I(t)\rho_B(0)\right\} &= \mathrm{Tr}_B\left\{H_I(t)\sum_i p_i\left|E_i\right\rangle\left\langle E_i\right|\right\} \\
&= \sum_i p_i \mathrm{Tr}_B\left\{H_I(t)\left|E_i\right\rangle\left\langle E_i\right|\right\} \\
&= \sum_i p_i\sum_k\left\langle E_k\right|\left\{H_I(t)\left|E_i\right\rangle\left\langle E_i\right|\right\}\left|E_k\right\rangle \\
&= \sum_i p_i\sum_k\left\langle E_k\right|\left\{H_I(t)\left|E_i\right\rangle\right\}\delta_{ik} \\
&= \sum_i p_i\left\langle E_i\right|H_I(t)\left|E_i\right\rangle \\
&= 0,
\end{aligned}$$

for we are assuming that $H_I(t)$ is linear in the creation and annihilation operators. Now we also have, for our previous interaction Hamiltonian example,

$$\begin{aligned}
\left[H_I(t),\rho_S(0)\right] &= \left[\hbar\sigma_z(t)\sum_s\left[g_s b_s\exp\left(-i\omega_s t\right)+g_s^* b_s^\dagger\exp\left(i\omega_s t\right)\right],\rho_S(0)\right] \\
&= \left[\hbar\sigma_z(t),\rho_S(0)\right]\sum_s\left[g_s b_s\exp\left(-i\omega_s t\right)+g_s^* b_s^\dagger\exp\left(i\omega_s t\right)\right]
\end{aligned}$$

and, again, because this commutator is linear in the creation and annihilation operators, we obtain

$$\mathrm{Tr}_B\left\{\left[H_I(t),\rho_S(0)\right]\rho_B(0)\right\} = 0.$$

Thus, the reduced density operator of the qubit evolves according to

$$\frac{d}{dt}\rho_{IS}(t) = -\frac{1}{\hbar^2}\mathrm{Tr}_B\left\{\int_0^t dt'\,\left[H_I(t),\left[H_I(t'),\rho_I(t')\right]\right]\right\}. \qquad (1)$$

If we look at the integrand in Eq. (2),

$$\frac{d}{dt}\rho_{IS}(t) = -\frac{1}{\hbar^2}\int_0^t dt' \, \mathrm{Tr}_B \left\{ \left[H_I(t), \left[H_I(t'), \rho_B(0)\rho_{IS}(t) \right] \right] \right\}, \tag{2}$$

we see that $\rho_B(0)$ is what appears in the Born approximation and not $\rho_B(t)$. Why? To answer this question, we are going to use the projection formalism that I explain below very briefly. Let us define a super operator, $\mathcal{P}$, defined by

$$\mathcal{P}A \equiv \rho_B(0)\,\mathrm{Tr}_B(A)$$

for any operator A. Let us also define the identity super operator, $\mathcal{I}$, as such that, for all operators A, gives:

$$\mathcal{I}A \equiv A.$$

Now we define the super operator $\mathcal{Q}$ thus:

$$\mathcal{Q} \equiv \mathcal{I} - \mathcal{P}.$$

We clearly see that, when acting on any operators, A, we have:

$$\begin{aligned}
\mathcal{P}^2 A &= \mathcal{P}\left[\rho_B(0)\,\mathrm{Tr}_B(A)\right] \\
&= \rho_B(0)\,\mathrm{Tr}_B\left[\rho_B(0)\,\mathrm{Tr}_B(A)\right] \\
&= \rho_B(0)\,\mathrm{Tr}_B(A)\,\mathrm{Tr}_B\left[\rho_B(0)\right] \\
&= \rho_B(0)\,\mathrm{Tr}_B(A) \\
&= \mathcal{P}A,
\end{aligned}$$

since

$$\mathrm{Tr}_B\left[\rho_B(0)\right] = 1.$$

Hence, the super operator $\mathcal{P}$ is a projection, since $\mathcal{P}^2 = \mathcal{P}$. The same property is true for $\mathcal{Q}$, namely,

$$\begin{aligned}
\mathcal{Q}^2 A &= \mathcal{Q}(\mathcal{I} - \mathcal{P})A \\
&= \mathcal{Q}(\mathcal{I}A - \mathcal{P}A) \\
&= \mathcal{Q}(A - \mathcal{P}A) \\
&= \mathcal{Q}A - \mathcal{Q}\mathcal{P}A \\
&= \mathcal{Q}A - (\mathcal{I} - \mathcal{P})\mathcal{P}A \\
&= \mathcal{Q}A - \left(\mathcal{I}\mathcal{P}A - \mathcal{P}^2 A\right) \\
&= \mathcal{Q}A - (\mathcal{P}A - \mathcal{P}A) \\
&= \mathcal{Q}A.
\end{aligned}$$

Therefore, we now can write our density matrix operator as

$$\begin{aligned}
\rho_I(t) &= (\mathcal{P} + \mathcal{Q})\,\rho_I(t) \\
&= \mathcal{P}\rho_I(t) + \mathcal{Q}\rho_I(t) \\
&= \rho_B(0)\,\rho_{IS}(t) + \mathcal{Q}\rho_I(t).
\end{aligned}$$

Thus,

$$\frac{d}{dt}\rho_I(t) = \rho_B(0)\frac{d}{dt}\rho_{IS}(t) + \frac{d}{dt}\mathcal{Q}\rho_I(t),$$

that is,

$$\frac{1}{i\hbar}[H_I(t), \rho_I(t)] = -\frac{1}{\hbar^2}\rho_B(0)\,\mathrm{Tr}_B\left\{\int_0^t dt'\ [H_I(t), [H_I(t'), \rho_I(t')]]\right\} + \frac{d}{dt}\mathcal{Q}\rho_I(t),$$

where here we recall Eq. (3),

$$\frac{d}{dt}\rho_{IS}(t) = -\frac{1}{\hbar^2}\mathrm{Tr}_B\left\{\int_0^t dt'\ [H_I(t), [H_I(t'), \rho_I(t')]]\right\}, \tag{3}$$

which we have just used above, or yet,

$$\begin{aligned}\frac{d}{dt}\mathcal{Q}\rho_I(t) &= \frac{1}{i\hbar}[H_I(t), \rho_I(t)] \\ &\quad + \frac{1}{\hbar^2}\rho_B(0)\,\mathrm{Tr}_B\left\{\int_0^t dt'\ [H_I(t), [H_I(t'), \rho_I(t')]]\right\}.\end{aligned} \tag{4}$$

Using

$$\rho_I(t') = \rho_B(0)\,\rho_{IS}(t') + \mathcal{Q}\rho_I(t')$$

in the integrand of Eq. (3), we obtain

$$\begin{aligned}\frac{d}{dt}\rho_{IS}(t) &= -\frac{1}{\hbar^2}\mathrm{Tr}_B\left\{\int_0^t dt'\ [H_I(t), [H_I(t'), \rho_B(0)\,\rho_{IS}(t')]]\right\} \\ &\quad - \frac{1}{\hbar^2}\mathrm{Tr}_B\left\{\int_0^t dt'\ [H_I(t), [H_I(t'), \mathcal{Q}\rho_I(t')]]\right\}.\end{aligned} \tag{5}$$

We can formally find the solution of Eq. (4) in the form:

$$\begin{aligned}\mathcal{Q}\rho_I(t') &= \mathcal{Q}\rho_I(0) + \frac{1}{i\hbar}\int_0^{t'} dt''\ [H_I(t''), \rho_I(t'')] \\ &\quad + \frac{1}{\hbar^2}\mathrm{Tr}_B\left\{\int_0^{t'} dt'' \int_0^{t''} dt'''\ [H_I(t''), [H_I(t'''), \rho_I(t''')]]\right\}.\end{aligned}$$

But then it becomes apparent that

$$\mathcal{Q}\rho_I(t') = \mathcal{Q}\rho_I(0) + \mathcal{O}(\lambda),$$

where λ is the strength magnitude of the coupling constants between the qubit and the bath. However,

$$\begin{aligned}\mathcal{Q}\rho_I(0) &= \mathcal{Q}[\rho_S(0)\,\rho_B(0)] \\ &= \rho_S(0)\,\rho_B(0) - \rho_B(0)\,\mathrm{Tr}_B[\rho_S(0)\,\rho_B(0)] \\ &= \rho_S(0)\,\rho_B(0) - \rho_B(0)\,\rho_S(0)\,\mathrm{Tr}_B[\rho_B(0)] \\ &= \rho_S(0)\,\rho_B(0) - \rho_S(0)\,\rho_B(0) \\ &= 0.\end{aligned}$$

Hence, updating our previous conclusion, we obtain

$$\mathcal{Q}\rho_I(t') = \mathcal{O}(\lambda).$$

Using this result into Eq. (5), we see that

$$\begin{aligned}\frac{d}{dt}\rho_{IS}(t) &= -\frac{1}{\hbar^2}\mathrm{Tr}_B\left\{\int_0^t dt'\ [H_I(t),[H_I(t'),\rho_B(0)\rho_{IS}(t')]]\right\}\\ &\quad -\frac{1}{\hbar^2}\mathrm{Tr}_B\left\{\int_0^t dt'\ [H_I(t),[H_I(t'),\mathcal{O}(\lambda)]]\right\}\\ &= -\frac{1}{\hbar^2}\mathrm{Tr}_B\left\{\int_0^t dt'\ [H_I(t),[H_I(t'),\rho_B(0)\rho_{IS}(t')]]\right\}\\ &\quad +\mathcal{O}\left(\lambda^3\right).\end{aligned}$$

Because we are not considering higher than the second order of λ in the master equation for the reduced density operator describing the qubit, we take this as our master equation:

$$\frac{d}{dt}\rho_{IS}(t) = -\frac{1}{\hbar^2}\mathrm{Tr}_B\left\{\int_0^t dt'\ [H_I(t),[H_I(t'),\rho_B(0)\rho_{IS}(t')]]\right\}.$$

From Eq. (6),

$$\rho_I(t_2) = \rho_I(t_1) + \frac{1}{i\hbar}\int_{t_1}^{t_2} dt'\ [H_I(t'),\rho_I(t')], \tag{6}$$

we derive that

$$\rho_I(t') = \rho_I(t) + \frac{1}{i\hbar}\int_t^{t'} dt''\ [H_I(t''),\rho_I(t'')],$$

namely,

$$\rho_I(t') = \rho_I(t) + \mathcal{O}(\lambda).$$

Thus,

$$\mathcal{P}\rho_I(t') = \mathcal{P}\rho_I(t) + \mathcal{O}(\lambda),$$

or yet,

$$\rho_B(0)\rho_{IS}(t') = \rho_B(0)\rho_{IS}(t) + \mathcal{O}(\lambda).$$

In the second-order Born approximation, therefore, we take our master equation to be time local:

$$\frac{d}{dt}\rho_{IS}(t) = -\frac{1}{\hbar^2}\mathrm{Tr}_B\left\{\int_0^t dt'\ [H_I(t),[H_I(t'),\rho_B(0)\rho_{IS}(t)]]\right\}. \tag{7}$$

It might look like we are cheating somehow. However, had we not multiplied by $\rho_B(0)$ the Tr_B appearing in the definition of $\mathcal{P}$, we would not have a projection for all operators A. Remember this

$$\begin{aligned}
\mathcal{P}^2 A &= \mathcal{P}\left[\rho_B(0)\,\mathrm{Tr}_B(A)\right] \\
&= \rho_B(0)\,\mathrm{Tr}_B\left[\rho_B(0)\,\mathrm{Tr}_B(A)\right] \\
&= \rho_B(0)\,\mathrm{Tr}_B(A)\,\mathrm{Tr}_B\left[\rho_B(0)\right] \\
&= \rho_B(0)\,\mathrm{Tr}_B(A) \\
&= \mathcal{P}A?
\end{aligned}$$

Hence, if we had used $\mathbb{I}_B$ instead of $\rho_B(0)$, we would have obtained $\mathcal{P}^2 A = \mathrm{Tr}_B(A)$ and, unless A happened to be a particular one, with $\mathrm{Tr}_B(A) = 1$, we would have ended up without a projection super operator $\mathcal{P}$. You could also argue that we could have used $\rho_B(\tau)$ for some instant of time τ and, true, we could. But then, by an argument analogous to what we have given above, we could show that

$$\rho_B(\tau) = \rho_B(0) + \mathcal{O}(\lambda)$$

and, again, we keep $\rho_B(0)$ and neglect the corrections of order λ, thus taking into account only terms of up to second order of λ in the master equation. Therefore, Eq. (7) is valid up to the second Born approximation.