

Master equation approximation

The master equation we start with is not the Lindblad equation, for, although we make the Born approximation of second order in the interaction coupling magnitude, we do not make the Markovian approximation. It is given, in the interaction picture, by

$$\frac{d}{dt}\rho_{IS}(t) = -\frac{1}{\hbar^2}\int_0^t dt' \text{Tr}_B \{[H_I(t), [H_I(t'), \rho_B(0)\rho_{IS}(t)]]\}, \quad (1)$$

which, as we can promptly see, is time-local as a consequence that we are considering the dynamics up to second order in the interaction, as stated above. Let us derive the master equation. We start from the Liouville equation for the interaction-picture density matrix. The density matrix used is the one that describes the composite system comprising the qubit together with the bath:

$$i\hbar\frac{d}{dt}\rho_I(t) = [H_I(t), \rho_I(t)].$$

The formal solution for this equation is given by

$$\rho_I(t_2) = \rho_I(t_1) + \frac{1}{i\hbar}\int_{t_1}^{t_2} dt' [H_I(t'), \rho_I(t')], \quad (2)$$

given two arbitrary instants of time t_1 and t_2 . Hence, it is also true that

$$\rho_I(t) = \rho_I(0) + \frac{1}{i\hbar}\int_0^t dt' [H_I(t'), \rho_I(t')]. \quad (3)$$

Using this in the right-hand side of the Liouville equation above, Eq. (1), we get

$$\begin{aligned} i\hbar\frac{d}{dt}\rho_I(t) &= [H_I(t), \rho_I(0)] \\ &\quad + \frac{1}{i\hbar}\left[H_I(t), \int_0^t dt' [H_I(t'), \rho_I(t')]\right], \end{aligned}$$

that is,

$$\begin{aligned} \frac{d}{dt}\rho_I(t) &= \frac{1}{i\hbar}[H_I(t), \rho_I(0)] \\ &\quad - \frac{1}{\hbar^2}\int_0^t dt' [H_I(t), [H_I(t'), \rho_I(t')]]. \end{aligned} \quad (4)$$

Tracing this equation over the bath degrees of freedom gives

$$\begin{aligned} \frac{d}{dt}\rho_{IS}(t) &= \frac{1}{i\hbar}\text{Tr}_B \{[H_I(t), \rho_I(0)]\} \\ &\quad - \frac{1}{\hbar^2}\text{Tr}_B \left\{ \int_0^t dt' [H_I(t), [H_I(t'), \rho_I(t')]] \right\}. \end{aligned}$$

As an example, let us consider the following interaction Hamiltonian in the interaction picture:

$$H_I(t) = \hbar\sigma_z(t) \sum_s [g_s b_s \exp(-i\omega_s t) + g_s^* b_s^\dagger \exp(i\omega_s t)],$$

which, as we see, is linear in the creation and annihilation operators. Starting from a factored state, namely,

$$\rho_I(0) = \rho_S(0) \rho_B(0),$$

where we assume that the boson bath is in a thermalized state, that is,

$$\rho_B(0) = \frac{\exp(-\beta \hbar \sum_s \omega_s b_s^\dagger b_s)}{Z},$$

$$\beta \equiv \frac{1}{k_B T},$$

and

$$Z \equiv \text{Tr}_B \left[\exp \left(-\beta \hbar \sum_s \omega_s b_s^\dagger b_s \right) \right].$$