Interação da radiação com a matéria

Estado coerente (continuação)

Mas e se quisermos usar esta nova representação no caso interagente? Como é que nossa hamiltoniana se transforma? É simples, escrevemos a equação de Schrödinger

$$i\hbar\frac{d}{dt}\left|\psi_{dip}\left(t\right)\right\rangle = \left(H_{at}+H_{rad}+H_{int}\right)\left|\psi_{dip}\left(t\right)\right\rangle ,$$

onde $\left|\psi_{dip}\left(t\right)\right\rangle$ é o estado do sistema todo na representação dipolar elétrica, já que não estamos mais usando a representação de Schrödinger original e usando essa notação não nos esqueceremos desse fato importante. Agora definimos, como acima, a nova representação, ou seja,

$$\left|\psi\left(t\right)\right\rangle \equiv U_{L}\left(t\right)\left|\psi_{dip}\left(t\right)\right\rangle ,$$

com

$$U_{L}\left(t\right) \equiv \exp\left[\alpha^{*}\exp\left(i\omega_{L}t\right)a_{L}-\alpha\exp\left(-i\omega_{L}t\right)a_{L}^{\dagger}\right], \tag{1}$$

onde aqui $\left|\psi\left(t\right)\right\rangle$ não vai ter mais subscritos, pois vamos manter esta representação até a conclusão deste tópico de interação da radiação com a matéria.

A equação de Schrödinger para $\left|\psi\left(t\right)\right\rangle$ fica

$$\begin{aligned}
i\hbar\frac{d}{dt}\left|\psi\left(t\right)\right\rangle &= i\hbar\left[\frac{dU_{L}\left(t\right)}{dt}\right]\left|\psi_{dip}\left(t\right)\right\rangle + i\hbar U_{L}\left(t\right)\frac{d}{dt}\left|\psi_{dip}\left(t\right)\right\rangle \\
&= i\hbar\left[\frac{dU_{L}\left(t\right)}{dt}\right]\left|\psi_{dip}\left(t\right)\right\rangle \\
&\quad +U_{L}\left(t\right)\left(H_{at}+H_{rad}+H_{int}\right)\left|\psi_{dip}\left(t\right)\right\rangle \\
&= i\hbar\left[\frac{dU_{L}\left(t\right)}{dt}\right]U_{L}^{\dagger}\left(t\right)U_{L}\left(t\right)\left|\psi_{dip}\left(t\right)\right\rangle \\
&\quad +U_{L}\left(t\right)\left(H_{at}+H_{rad}+H_{int}\right)U_{L}^{\dagger}\left(t\right)U_{L}\left(t\right)\left|\psi_{dip}\left(t\right)\right\rangle \\
&= i\hbar\left[\frac{dU_{L}\left(t\right)}{dt}\right]U_{L}^{\dagger}\left(t\right)\left|\psi\left(t\right)\right\rangle \\
&\quad +U_{L}\left(t\right)\left(H_{at}+H_{rad}+H_{int}\right)U_{L}^{\dagger}\left(t\right)\left|\psi\left(t\right)\right\rangle .
\end{aligned} \tag{2}$$

Comecemos calculando a derivada de $U_{L}\left(t\right)$ notando a Eq. (3),

$$\exp\left(\alpha a^{\dagger}\right)\exp\left(-\alpha^{*}a\right) = \exp\left(\frac{\left|\alpha\right|^{2}}{2}\right)\exp\left(\alpha a^{\dagger}-\alpha^{*}a\right), \tag{3}$$

isto é,

$$\begin{aligned}
\frac{dU_{L}\left(t\right)}{dt} &= \frac{d}{dt}\exp\left[\alpha^{*}\exp\left(i\omega_{L}t\right)a_{L}-\alpha\exp\left(-i\omega_{L}t\right)a_{L}^{\dagger}\right] \\
&= \exp\left(-\frac{\left|\alpha\right|^{2}}{2}\right)\frac{d}{dt}\left\{\exp\left[-\alpha\exp\left(-i\omega_{L}t\right)a_{L}^{\dagger}\right]\exp\left[\alpha^{*}\exp\left(i\omega_{L}t\right)a_{L}\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \exp\left(-\frac{|\alpha|^2}{2}\right)\left\{\frac{d}{dt}\exp\left[-\alpha\exp\left(-i\omega_L t\right)a_L^\dagger\right]\right\}\exp\left[\alpha^*\exp\left(i\omega_L t\right)a_L\right] \\
&\quad +\exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left[-\alpha\exp\left(-i\omega_L t\right)a_L^\dagger\right]\frac{d}{dt}\exp\left[\alpha^*\exp\left(i\omega_L t\right)a_L\right] \\
&= i\omega_L\alpha\exp\left(-i\omega_L t\right)a_L^\dagger U_L\left(t\right)+i\omega_L\alpha^*\exp\left(i\omega_L t\right)\exp\left(-\frac{|\alpha|^2}{2}\right) \\
&\quad \times\exp\left[-\alpha\exp\left(-i\omega_L t\right)a_L^\dagger\right]a_L\exp\left[\alpha^*\exp\left(i\omega_L t\right)a_L\right].
\end{aligned}
$$

Aí podemos usar a Eq. (4),

$$
\exp\left(\alpha a^\dagger\right)a = \left(a-\alpha\right)\exp\left(\alpha a^\dagger\right), \tag{4}
$$

e obter

$$
\exp\left[-\alpha\exp\left(-i\omega_L t\right)a_L^\dagger\right]a_L = \left[a_L+\alpha\exp\left(-i\omega_L t\right)\right]\exp\left[-\alpha\exp\left(-i\omega_L t\right)a_L^\dagger\right].
$$

Com esta igualdade,

$$
\begin{aligned}
\frac{dU_L\left(t\right)}{dt} &= i\omega_L\alpha\exp\left(-i\omega_L t\right)a_L^\dagger U_L\left(t\right)+i\omega_L\alpha^*\exp\left(i\omega_L t\right) \\
&\quad \times\left[a_L+\alpha\exp\left(-i\omega_L t\right)\right]U_L\left(t\right),
\end{aligned}
$$

ou ainda,

$$
\frac{dU_L\left(t\right)}{dt} = \left[i\omega_L\alpha\exp\left(-i\omega_L t\right)a_L^\dagger+i\omega_L\alpha^*\exp\left(i\omega_L t\right)a_L+i\omega_L\left|\alpha\right|^2\right]U_L\left(t\right). \tag{5}
$$

A seguir temos que

$$
\begin{aligned}
U_L\left(t\right)\left(H_{at}+H_{rad}+H_{int}\right)U_L^\dagger\left(t\right) &= U_L\left(t\right)H_{at}U_L^\dagger\left(t\right) \\
&\quad +U_L\left(t\right)H_{rad}U_L^\dagger\left(t\right)+U_L\left(t\right)H_{int}U_L^\dagger\left(t\right).
\end{aligned} \tag{6}
$$

Mas, claramente pelo fato de que H_{at} comuta com os operadores a_L e $a_L^\dagger$, então

$$
U_L\left(t\right)H_{at}U_L^\dagger\left(t\right) = H_{at}. \tag{7}
$$

Agora precisamos apenas calcular como a_L se transforma, ou seja,

$$
\begin{aligned}
U_L\left(t\right)a_L U_L^\dagger\left(t\right) &= \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left[-\alpha\exp\left(-i\omega_L t\right)a_L^\dagger\right] \\
&\quad \times\exp\left[\alpha^*\exp\left(i\omega_L t\right)a_L\right]a_L U_L^\dagger\left(t\right),
\end{aligned} \tag{8}
$$

onde já usamos que, por causa da Eq. (3), podemos escrever

$$
U_L\left(t\right) = \exp\left(-\frac{|\alpha|^2}{2}\right)\exp\left[-\alpha\exp\left(-i\omega_L t\right)a_L^\dagger\right]\exp\left[\alpha^*\exp\left(i\omega_L t\right)a_L\right].
$$

Agora, para ficar mais conveniente, podemos inverter a Eq. (3) primeiro notando que

$$\left[\exp\left(\alpha a^{\dagger}-\alpha^{*}a\right)\right]^{-1} = \exp\left(\frac{|\alpha|^{2}}{2}\right)\left[\exp\left(\alpha a^{\dagger}\right)\exp\left(-\alpha^{*}a\right)\right]^{-1}$$

e, a seguir, que

$$\left[\exp\left(\alpha a^{\dagger}\right)\exp\left(-\alpha^{*}a\right)\right]^{-1} = \exp\left(\alpha^{*}a\right)\exp\left(-\alpha a^{\dagger}\right).$$

Então,

$$\left[\exp\left(\alpha a^{\dagger}-\alpha^{*}a\right)\right]^{-1} = \exp\left(\frac{|\alpha|^{2}}{2}\right)\exp\left(\alpha^{*}a\right)\exp\left(-\alpha a^{\dagger}\right)$$

e, portanto,

$$\exp\left(-\alpha a^{\dagger}+\alpha^{*}a\right) = \exp\left(\frac{|\alpha|^{2}}{2}\right)\exp\left(\alpha^{*}a\right)\exp\left(-\alpha a^{\dagger}\right),$$

ou ainda,

$$\exp\left(-\alpha^{*}a+\alpha a^{\dagger}\right) = \exp\left(\frac{|\alpha|^{2}}{2}\right)\exp\left(-\alpha^{*}a\right)\exp\left(\alpha a^{\dagger}\right). \tag{9}$$

Como da Eq. (1) decorre que

$$U_{L}^{\dagger}(t) = \exp\left[-\alpha^{*}\exp\left(i\omega_{L}t\right)a_{L}+\alpha\exp\left(-i\omega_{L}t\right)a_{L}^{\dagger}\right],$$

Usando a Eq. (9) obtemos

$$U_{L}^{\dagger}(t) = \exp\left(\frac{|\alpha|^{2}}{2}\right)\exp\left[-\alpha^{*}\exp\left(i\omega_{L}t\right)a_{L}\right]\exp\left[\alpha\exp\left(-i\omega_{L}t\right)a_{L}^{\dagger}\right].$$

Substituindo esta expressão de $U_{L}^{\dagger}(t)$ na Eq. (8) dá

$$\begin{aligned} U_{L}(t)\,a_{L}U_{L}^{\dagger}(t) = {} & \exp\left(-\frac{|\alpha|^{2}}{2}\right)\exp\left[-\alpha\exp\left(-i\omega_{L}t\right)a_{L}^{\dagger}\right] \\ & \times\exp\left[\alpha^{*}\exp\left(i\omega_{L}t\right)a_{L}\right]a_{L}\exp\left[-\alpha^{*}\exp\left(i\omega_{L}t\right)a_{L}\right] \\ & \times\exp\left[\alpha\exp\left(-i\omega_{L}t\right)a_{L}^{\dagger}\right]\exp\left(\frac{|\alpha|^{2}}{2}\right), \end{aligned}$$

isto é,

$$U_{L}(t)\,a_{L}U_{L}^{\dagger}(t) = \exp\left[-\alpha\exp\left(-i\omega_{L}t\right)a_{L}^{\dagger}\right]a_{L}\exp\left[\alpha\exp\left(-i\omega_{L}t\right)a_{L}^{\dagger}\right].$$

Se agora voltarmos à sexta aula, veremos a dedução de que

$$\exp\left(-i\beta a_{\lambda,s}^{\dagger}\right) a_{\lambda,s} \exp\left(i\beta a_{\lambda,s}^{\dagger}\right) = a_{\lambda,s} + i\beta$$

e, assim, basta fazer

$$\beta = -i\alpha \exp\left(-i\omega_L t\right)$$

e

$$(\lambda, s) \leftrightarrow L$$

para obtermos

$$U_L(t)\, a_L U_L^{\dagger}(t) = a_L + \alpha \exp\left(-i\omega_L t\right). \tag{10}$$

Voltando à Eq. (6), podemos agora escrever os termos que faltam à luz da Eq. (10), notando que

$$\begin{aligned}
U_L(t)\, H_{rad} U_L^{\dagger}(t) &= \sum_{(\lambda,s)\neq L} \hbar\omega_s \left(a_{\lambda,s}^{\dagger} a_{\lambda,s} + \frac{1}{2}\right) \\
&\quad + \hbar\omega_L \left\{\left[a_L^{\dagger} + \alpha^* \exp\left(i\omega_L t\right)\right]\left[a_L + \alpha \exp\left(-i\omega_L t\right)\right] + \frac{1}{2}\right\} \\
&= \sum_{(\lambda,s)\neq L} \hbar\omega_s \left(a_{\lambda,s}^{\dagger} a_{\lambda,s} + \frac{1}{2}\right) + \frac{1}{2}\hbar\omega_L \\
&\quad + \hbar\omega_L \left[a_L^{\dagger} + \alpha^* \exp\left(i\omega_L t\right)\right]\left[a_L + \alpha \exp\left(-i\omega_L t\right)\right] \\
&= \sum_{(\lambda,s)\neq L} \hbar\omega_s \left(a_{\lambda,s}^{\dagger} a_{\lambda,s} + \frac{1}{2}\right) + \hbar\omega_L \left(a_L^{\dagger} a_L + \frac{1}{2}\right) \\
&\quad + \hbar\omega_L \left[\alpha \exp\left(-i\omega_L t\right) a_L^{\dagger} + \alpha^* \exp\left(i\omega_L t\right) a_L + |\alpha|^2\right],
\end{aligned}$$

ou seja,

$$\begin{aligned}
U_L(t)\, H_{rad} U_L^{\dagger}(t) &= H_{rad} + \hbar\omega_L \alpha \exp\left(-i\omega_L t\right) a_L^{\dagger} \\
&\quad + \hbar\omega_L \alpha^* \exp\left(i\omega_L t\right) a_L + \hbar\omega_L |\alpha|^2 .
\end{aligned} \tag{11}$$

Também utilizando a Eq. (10) escrevemos

$$\begin{aligned}
U_L(t)\, H_{int} U_L^{\dagger}(t) &= -q\mathbf{r} \cdot \left[\sum_{(\lambda,s)\neq L} i\sqrt{\frac{2\pi\hbar\omega_s}{V}}\, \hat{\boldsymbol{\epsilon}}_{\lambda,s} \left(a_{\lambda,s} - a_{\lambda,s}^{\dagger}\right)\right] \\
&\quad - q\mathbf{r} \cdot \left\{ i\sqrt{\frac{2\pi\hbar\omega_L}{V}}\, \hat{\boldsymbol{\epsilon}}_L \left[a_L + \alpha \exp\left(-i\omega_L t\right)\right.\right. \\
&\quad \left.\left. - a_L^{\dagger} - \alpha^* \exp\left(i\omega_L t\right)\right]\right\},
\end{aligned}$$

isto é,

$$
\begin{aligned}
U_L(t)\, H_{int} U_L^{\dagger}(t) &= -q\mathbf{r}\cdot\left[\sum_{\lambda,s} i\sqrt{\frac{2\pi\hbar\omega_s}{V}}\hat{\boldsymbol{\epsilon}}_{\lambda,s}\left(a_{\lambda,s}-a_{\lambda,s}^{\dagger}\right)\right] \\
&\quad -q\mathbf{r}\cdot\left\{i\sqrt{\frac{2\pi\hbar\omega_L}{V}}\hat{\boldsymbol{\epsilon}}_L\left[\alpha\exp\left(-i\omega_L t\right)-\alpha^{*}\exp\left(i\omega_L t\right)\right]\right\}.
\end{aligned}
$$

Como definimos na sexta aula, o campo elétrico quântico é dado por

$$
\mathbf{E}(\mathbf{0}) = \sum_{\lambda,s} i\sqrt{\frac{2\pi\hbar\omega_s}{V}}\hat{\boldsymbol{\epsilon}}_{\lambda,s}\left(a_{\lambda,s}-a_{\lambda,s}^{\dagger}\right)
$$

e agora, nesta nova representação, temos também um campo elétrico clássico:

$$
\mathbf{E}_{cl}(t) \equiv i\sqrt{\frac{2\pi\hbar\omega_L}{V}}\hat{\boldsymbol{\epsilon}}_L\left[\alpha\exp\left(-i\omega_L t\right)-\alpha^{*}\exp\left(i\omega_L t\right)\right], \tag{12}
$$

que vai ser dependente do tempo. Em termos desses campos, temos a nova hamiltoniana de interação escrita como

$$
U_L(t)\, H_{int} U_L^{\dagger}(t) = -q\mathbf{r}\cdot\left[\mathbf{E}(\mathbf{0})+\mathbf{E}_{cl}(t)\right]. \tag{13}
$$

Em resumo, substituindo as Eqs. (7), (11) e (13) na Eq. (6), obtemos

$$
\begin{aligned}
U_L(t)\left(H_{at}+H_{rad}+H_{int}\right) U_L^{\dagger}(t) &= H_{at}+H_{rad}+\hbar\omega_L\alpha\exp\left(-i\omega_L t\right) a_L^{\dagger} \\
&\quad +\hbar\omega_L\alpha^{*}\exp\left(i\omega_L t\right) a_L+\hbar\omega_L\left|\alpha\right|^2 \\
&\quad -q\mathbf{r}\cdot\left[\mathbf{E}(\mathbf{0})+\mathbf{E}_{cl}(t)\right].
\end{aligned} \tag{14}
$$

Finalmente, voltando a considerar a Eq. (2), após substituição das Eqs. (5) e (14), vem

$$
\begin{aligned}
i\hbar\frac{d}{dt}\left|\psi(t)\right\rangle &= i\hbar\left[i\omega_L\alpha\exp\left(-i\omega_L t\right) a_L^{\dagger}+i\omega_L\alpha^{*}\exp\left(i\omega_L t\right) a_L+i\omega_L\left|\alpha\right|^2\right] U_L(t)\, U_L^{\dagger}(t)\left|\psi(t)\right\rangle \\
&\quad H_{at}\left|\psi(t)\right\rangle+H_{rad}\left|\psi(t)\right\rangle+\hbar\omega_L\alpha\exp\left(-i\omega_L t\right) a_L^{\dagger}\left|\psi(t)\right\rangle \\
&\quad +\hbar\omega_L\alpha^{*}\exp\left(i\omega_L t\right) a_L\left|\psi(t)\right\rangle+\hbar\omega_L\left|\alpha\right|^2\left|\psi(t)\right\rangle \\
&\quad -q\mathbf{r}\cdot\left[\mathbf{E}(\mathbf{0})+\mathbf{E}_{cl}(t)\right]\left|\psi(t)\right\rangle,
\end{aligned}
$$

ou seja,

$$
\begin{aligned}
i\hbar\frac{d}{dt}\left|\psi(t)\right\rangle &= -\hbar\omega_L\alpha\exp\left(-i\omega_L t\right) a_L^{\dagger}\left|\psi(t)\right\rangle-\hbar\omega_L\alpha^{*}\exp\left(i\omega_L t\right) a_L\left|\psi(t)\right\rangle \\
&\quad -\hbar\omega_L\left|\alpha\right|^2\left|\psi(t)\right\rangle+H_{at}\left|\psi(t)\right\rangle \\
&\quad +H_{rad}\left|\psi(t)\right\rangle+\hbar\omega_L\alpha\exp\left(-i\omega_L t\right) a_L^{\dagger}\left|\psi(t)\right\rangle \\
&\quad +\hbar\omega_L\alpha^{*}\exp\left(i\omega_L t\right) a_L\left|\psi(t)\right\rangle+\hbar\omega_L\left|\alpha\right|^2\left|\psi(t)\right\rangle \\
&\quad -q\mathbf{r}\cdot\left[\mathbf{E}(\mathbf{0})+\mathbf{E}_{cl}(t)\right]\left|\psi(t)\right\rangle,
\end{aligned}
$$

ou ainda,

$$i\hbar\frac{d}{dt}\left|\psi\left(t\right)\right\rangle = \left\{H_{at}+H_{rad}-q\mathbf{r}\cdot\left[\mathbf{E}\left(\mathbf{0}\right)+\mathbf{E}_{cl}\left(t\right)\right]\right\}\left|\psi\left(t\right)\right\rangle .$$

Portanto, a nova hamiltoniana para nosso modelo de interação da radiação com a matéria fica

$$H\left(t\right) \equiv H_{at}+H_{rad}-q\mathbf{r}\cdot\left[\mathbf{E}\left(\mathbf{0}\right)+\mathbf{E}_{cl}\left(t\right)\right]. \tag{15}$$

O estado inicial do sistema total, isto é, do campo eletromagnético e do átomo, é dado por

$$\left|\psi\left(0\right)\right\rangle = \left|\phi_{at}\left(0\right)\right\rangle\otimes\left|\mathrm{vac}\right\rangle . \tag{16}$$

A nova hamiltoniana, Eq. (15), depende explicitamente do tempo nesta representação. Em outros termos, a nova hamiltoniana é não autônoma, ou seja, nesta representação é como se houvesse um agente externo ao sistema introduzindo um campo externo dependente do tempo de forma orquestrada para manipular o comportamento do sistema. É como se tivéssemos deixado o laser do lado de fora do sistema que só consiste de um átomo e o vácuo eletromagnético. Um exemplo análogo em mecânica clássica seria o da hamiltoniana de uma partícula que se move sem atrito sobre a superfície de uma bexiga que está sendo inflada e, portanto, é uma função dependente do tempo.