Logo, no caso que tratamos nessas aulas de sobre nossos passos, teremos:

$$\begin{aligned} W(x) &= \exp\left[-\int dx \left(1-\frac{2}{x}\right)\right] \\ &= \exp(-x+2\ln x) \\ &= \exp(-x)\exp(2\ln x) \\ &= \exp(-x)\exp\left(\ln x^2\right) \\ &= x^2\exp(-x) \\ &= y_1(x). \end{aligned}$$

Desta forma,

$$\begin{aligned} y_2^W(x) &= y_1(x)\int dx\,\frac{y_1(x)}{[y_1(x)]^2} \\ &= y_1(x)\int dx\,\frac{1}{y_1(x)} \\ &= x^2\exp(-x)\int dx\,\frac{\exp(x)}{x^2}. \end{aligned}$$

Aqui estamos usando $y_2^W(x)$ para explicitar que essa é a segunda solução obtida através do método do Wronskiano. Vemos que essa solução não se parece com a que obtivemos pelo método de Frobenius, que é dada por:

$$y_2^F(x) = -x^2\exp(-x)\ln x + x + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(\sum_{k=1}^{n}\frac{1}{k}\right)x^{n+2}.$$

Aqui, usamos $y_2^F(x)$ para indicar que essa é a segunda solução obtida pelo método de Frobenius. Precisamos, então, mostrar que podemos transformar o que obtivemos acima para $y_2^W(x)$ no resultado $y_2^F(x)$.

Olhando a forma da solução $y_2^W(x)$, podemos fazer aparecer o logaritmo notando o seguinte:

$$\begin{aligned} \int dx\,\frac{\exp(x)}{x^2} &= \int dx \sum_{n=0}^{\infty}\frac{1}{n!}x^{n-2} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\int dx\,x^{n-2} \\ &= \int dx\,x^{-2} + \int dx\,x^{-1} + \sum_{n=2}^{\infty}\frac{1}{n!}\int dx\,x^{n-2} \\ &= -\frac{1}{x} + \ln x + \sum_{n=2}^{\infty}\frac{1}{n!}\int dx\,x^{n-2}. \end{aligned}$$

Portanto,

$$y_2^W(x) = -x\exp(-x) + x^2\exp(-x)\ln x + x^2\exp(-x)\sum_{n=2}^{\infty}\frac{1}{n!}\int dx\,x^{n-2}$$

$$
\begin{aligned}
&= x^2\exp(-x)\ln x - x\exp(-x) + x^2\exp(-x)\sum_{n=2}^{\infty}\frac{1}{n!}\int dx\, x^{n-2} \\
&= x^2\exp(-x)\ln x - x\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n + x^2\exp(-x)\sum_{n=2}^{\infty}\frac{1}{n!}\int dx\, x^{n-2} \\
&= x^2\exp(-x)\ln x - x - x\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}x^n + x^2\exp(-x)\sum_{n=2}^{\infty}\frac{1}{n!}\int dx\, x^{n-2}.
\end{aligned}
$$

Agora podemos usar uma “mágica” que é notar que

$$
\begin{aligned}
x\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}x^n &= -x^2\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}x^n \\
&= -x^2\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}x^n - x^2\sum_{n=0}^{\infty}\frac{n(-1)^n}{(n+1)!}x^n + x^2\sum_{n=0}^{\infty}\frac{n(-1)^n}{(n+1)!}x^n \\
&= -x^2\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n + x^2\sum_{n=0}^{\infty}\frac{n(-1)^n}{(n+1)!}x^n \\
&= -x^2\exp(-x) + x^2\sum_{n=0}^{\infty}\frac{n(-1)^n}{(n+1)!}x^n \\
&= -y_1(x) + x^2\sum_{n=1}^{\infty}\frac{n(-1)^n}{(n+1)!}x^n.
\end{aligned}
$$

Logo,

$$
\begin{aligned}
y_2^W(x) &= x^2\exp(-x)\ln x - x + y_1(x) - x^2\sum_{n=1}^{\infty}\frac{n(-1)^n}{(n+1)!}x^n + x^2\exp(-x)\sum_{n=2}^{\infty}\frac{1}{n!}\int dx\, x^{n-2} \\
&= x^2\exp(-x)\ln x - x + y_1(x) + \sum_{n=2}^{\infty}\frac{(n-1)(-1)^n}{n!}x^{n+1} + x^2\exp(-x)\sum_{n=2}^{\infty}\frac{1}{n!}\int dx\, x^{n-2} \\
&= x^2\exp(-x)\ln x - x + y_1(x) + \sum_{n=2}^{\infty}\frac{(n-1)(-1)^n}{n!}x^{n+1} + \exp(-x)\sum_{n=2}^{\infty}\frac{1}{n!}\frac{x^{n+1}}{n-1}.
\end{aligned}
$$

O que é preciso verificar é que

$$
f(x) + g(x) = 0,
$$

onde definimos as funções:

$$
f(x) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(\sum_{k=1}^{n}\frac{1}{k}\right)x^{n+2}
$$

e

$$
g(x) = \sum_{n=2}^{\infty}\frac{(n-1)(-1)^n}{n!}x^{n+1} + \exp(-x)\sum_{n=2}^{\infty}\frac{1}{n!}\frac{x^{n+1}}{n-1}.
$$

De acordo com o programa Mathematica, temos a figura abaixo.

```
In[3]:= a = Sum[(n - 1) (-1)^n x^(n+1)/n!, {n, 2, Infinity}]
```

Out[3]= $e^{-x}\,(-1 + e^{x} - x)\,x$

```
In[4]:= b = Exp[-x] Sum[x^(n+1)/((n - 1) n!), {n, 2, Infinity}]
```

Out[4]= $e^{-x}\,x\,\left(1 - e^{x} + x - \text{EulerGamma}\,x - x\,\text{Gamma}[0, -x] - x\,\text{Log}[-x]\right)$

```
In[18]:= Series[a + b, {x, 0, 12}, Assumptions → x > 0]
```

Out[18]= $x^3 - \frac{3x^4}{4} + \frac{11x^5}{36} - \frac{25x^6}{288} + \frac{137x^7}{7200} - \frac{49x^8}{14\,400} + \frac{121x^9}{235\,200} - \frac{761x^{10}}{11\,289\,600} + \frac{7129x^{11}}{914\,457\,600} - \frac{7381x^{12}}{9\,144\,576\,000} +$

```
In[17]:= Sum[((-1)^n/n!) Sum[1/k, {k, 1, n}] x^(n+2), {n, 1, 10}]
```

Out[17]= $-x^3 + \frac{3x^4}{4} - \frac{11x^5}{36} + \frac{25x^6}{288} - \frac{137x^7}{7200} + \frac{49x^8}{14\,400} - \frac{121x^9}{235\,200} + \frac{761x^{10}}{11\,289\,600} - \frac{7129x^{11}}{914\,457\,600} + \frac{7381x^{12}}{9\,144\,576\,000}$