O que faríamos se não tivéssemos como somar a série em y_1 e obter $x^2 \exp(-x)$

Nesse caso, teríamos que calcular

$$\mathcal{L}(y_1 \ln x) \quad = \quad \mathcal{L}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+2} \ln x\right).$$

Já sabemos que

$$\mathcal{L} y_1 \quad = \quad 0.$$

Portanto,

$$\begin{aligned}
\mathcal{L}(y_1 \ln x) &= \frac{\partial^2}{\partial x^2}(y_1 \ln x) + \left(1 - \frac{2}{x}\right)\frac{\partial}{\partial x}(y_1 \ln x) + \frac{2}{x^2}(y_1 \ln x) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial y_1}{\partial x}\ln x + \frac{y_1}{x}\right) + \left(1 - \frac{2}{x}\right)\left(\frac{\partial y_1}{\partial x}\ln x + \frac{y_1}{x}\right) + \frac{2}{x^2}(y_1 \ln x) \\
&= \frac{\partial^2 y_1}{\partial x^2}\ln x + \frac{1}{x}\frac{\partial y_1}{\partial x} + \frac{\partial}{\partial x}\left(\frac{y_1}{x}\right) + \left(1 - \frac{2}{x}\right)\left(\frac{\partial y_1}{\partial x}\ln x + \frac{y_1}{x}\right) + \frac{2}{x^2}(y_1 \ln x) \\
&= \left[\frac{\partial^2 y_1}{\partial x^2} + \left(1 - \frac{2}{x}\right)\frac{\partial y_1}{\partial x} + \frac{2}{x^2}y_1\right]\ln x + \frac{2}{x}\frac{\partial y_1}{\partial x} - \frac{y_1}{x^2} + \left(1 - \frac{2}{x}\right)\frac{y_1}{x} \\
&= (\mathcal{L} y_1)\ln x + \frac{2}{x}\frac{\partial y_1}{\partial x} - \frac{y_1}{x^2} + \left(1 - \frac{2}{x}\right)\frac{y_1}{x} \\
&= 0 + \frac{2}{x}\frac{\partial y_1}{\partial x} + \frac{y_1}{x} - 3\frac{y_1}{x^2} \\
&= \frac{2}{x}\frac{\partial y_1}{\partial x} + \frac{y_1}{x} - 3\frac{y_1}{x^2}.
\end{aligned}$$

Supondo que só sabemos a forma em série de y_1, escrevemos

$$y_1 \quad = \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+2}$$

e calculamos

$$\frac{\partial y_1}{\partial x} \quad = \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(n+2)\, x^{n+1}.$$

Com isso, obtemos

$$\begin{aligned}
\mathcal{L}(y_1 \ln x) &= \frac{2}{x}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(n+2)\, x^{n+1} + \frac{1}{x}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+2} - \frac{3}{x^2}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+2} \\
&= 2\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(n+2)\, x^{n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+1} - 3\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(2n+4-3)\, x^{n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (2n+1)\, x^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+1} \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} (2n+1)\, x^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+1} \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} (2n+1)\, x^n - \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} x^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} (2n+1)\, x^n - \sum_{n=1}^{\infty} \frac{(-1)^n}{n\,(n-1)!} n x^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} (2n+1)\, x^n - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} n x^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} (2n+1-n)\, x^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} (n+1)\, x^n .
\end{aligned}
$$

Só para confirmar que esse resultado é como o que obtivemos acima, agora novamente supondo conhecida a série de $\exp(-x)$, é fácil verificarmos que, de fato,

$$
\begin{aligned}
1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} (n+1)\, x^n &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^n + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} n x^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} x^n \\
&= \exp(-x) - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+1} \\
&= \exp(-x) - x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \\
&= (1-x) \exp(-x) ,
\end{aligned}
$$

como tínhamos obtido anteriormente.