Convergência da série de Fourier

Aqui vamos fazer algo similar ao que o problema 10.18 do livro do Boyce et al. faz. Suponhamos que $f(\theta)$ e sua primeira derivada sejam contínuas, pelo menos em pedaços, sobre os reais. Também, como vimos,

$$f(\theta) = \sum_{n=-\infty}^{+\infty} c_n \exp(in\theta) \tag{1}$$

e

$$c_n \equiv \frac{1}{2\pi}\int_0^{2\pi} d\varphi \exp(-in\varphi) f(\varphi), \tag{2}$$

Então, usando a Eq. (2) obtemos

$$\begin{aligned}
nc_n &= \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, n \exp(-in\varphi) f(\varphi) \\
&= \frac{i}{2\pi}\int_0^{2\pi} d\varphi \left[\frac{d}{d\varphi}\exp(-in\varphi)\right] f(\varphi) \\
&= \frac{i}{2\pi}\int_0^{2\pi} d\varphi \frac{d}{d\varphi}\left[\exp(-in\varphi) f(\varphi)\right] - \frac{i}{2\pi}\int_0^{2\pi} d\varphi \exp(-in\varphi)\frac{d}{d\varphi} f(\varphi) \\
&= \frac{i}{2\pi}\exp(-in\varphi) f(\varphi)\Big|_0^{2\pi} - \frac{i}{2\pi}\int_0^{2\pi} d\varphi \exp(-in\varphi) f'(\varphi) \\
&= \frac{i}{2\pi}\left[\exp(-in2\pi) f(2\pi) - \exp(-in\times 0) f(0)\right] \\
&\quad -\frac{i}{2\pi}\int_0^{2\pi} d\varphi \exp(-in\varphi) f'(\varphi) \\
&= -\frac{i}{2\pi}\int_0^{2\pi} d\varphi \exp(-in\varphi) f'(\varphi).
\end{aligned}$$

Mas, então,

$$\begin{aligned}
|nc_n| &= \left|\left(-\frac{i}{2\pi}\right)\left[\int_0^{2\pi} d\varphi \exp(-in\varphi) f'(\varphi)\right]\right| \\
&= \left|-\frac{i}{2\pi}\right|\left|\int_0^{2\pi} d\varphi \exp(-in\varphi) f'(\varphi)\right| \\
&= \frac{1}{2\pi}\left|\int_0^{2\pi} d\varphi \exp(-in\varphi) f'(\varphi)\right| \\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi} \left|d\varphi \exp(-in\varphi) f'(\varphi)\right| \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\varphi \left|\exp(-in\varphi)\right| \left|f'(\varphi)\right| \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\varphi \left|f'(\varphi)\right|.
\end{aligned}$$

Logo, como

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi\,\left|f'(\varphi)\right| \quad < \quad \infty,$$

segue que nc_n é limitado quando $n \to \infty$. Isto é, como $\frac{1}{2\pi}\int_0^{2\pi} d\varphi\,\left|f'(\varphi)\right|$ é um número que não depende de n e é finito porque $f'(\varphi)$ é contínua, segue que quando n cresce ilimitadamente, então nc_n não diverge:

$$\begin{aligned}\lim_{n\to\infty}\left|c_n\right| &\leqslant \lim_{n\to\infty}\frac{\frac{1}{2\pi}\int_0^{2\pi} d\varphi\,\left|f'(\varphi)\right|}{n}\\ &= 0.\end{aligned}$$

Vamos supor também que $f(\theta)$ é contínua sobre reais e $f'(\theta)$ e a segunda derivada $f''(\theta)$ também são contínuas, pelo menos em pedaços, sobre os reais. Agora, ainda usando a Eq. (2), temos que

$$\begin{aligned}n^2 c_n &= \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, n^2 \exp(-in\varphi)\, f(\varphi)\\ &= \frac{in}{2\pi}\int_0^{2\pi} d\varphi\,\left[\frac{d}{d\varphi}\exp(-in\varphi)\right] f(\varphi)\\ &= \frac{in}{2\pi}\int_0^{2\pi} d\varphi\,\frac{d}{d\varphi}\left[\exp(-in\varphi)\, f(\varphi)\right] - \frac{i}{2\pi}\int_0^{2\pi} d\varphi\, n\exp(-in\varphi)\, f'(\varphi)\\ &= \frac{in}{2\pi}\exp(-in\varphi)\, f(\varphi)\Big|_0^{2\pi} - \frac{i}{2\pi}\int_0^{2\pi} d\varphi\, n\exp(-in\varphi)\, f'(\varphi)\\ &= -\frac{i}{2\pi}\int_0^{2\pi} d\varphi\, n\exp(-in\varphi)\, f'(\varphi).\end{aligned}$$

Mas, iterando o que acabamos de fazer, obtemos

$$\begin{aligned}n^2 c_n &= \frac{1}{2\pi}\int_0^{2\pi} d\varphi\,\left[\frac{d}{d\varphi}\exp(-in\varphi)\right] f'(\varphi)\\ &= \frac{1}{2\pi}\int_0^{2\pi} d\varphi\,\frac{d}{d\varphi}\left[\exp(-in\varphi)\, f'(\varphi)\right] - \frac{1}{2\pi}\int_0^{2\pi} d\varphi\,\exp(-in\varphi)\, f''(\varphi)\\ &= \frac{1}{2\pi}\exp(-in\varphi)\, f'(\varphi)\Big|_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} d\varphi\,\exp(-in\varphi)\, f''(\varphi).\end{aligned}$$

Quando uma função é periódica,

$$f(\theta + 2\pi) \quad = \quad f(\theta),$$

podemos tomar a derivada e voilà:

$$f'(\theta + 2\pi) \quad = \quad f'(\theta),$$

isto é, sua derivada é periódica também! Assim,

$$n^2 c_n = -\frac{1}{2\pi}\int_0^{2\pi} d\varphi \exp\left(-in\varphi\right) f''\left(\varphi\right)$$

e, portanto,

$$\begin{aligned}
\left|n^2 c_n\right| &= \left|-\frac{1}{2\pi}\int_0^{2\pi} d\varphi \exp\left(-in\varphi\right) f''\left(\varphi\right)\right| \\
&= \frac{1}{2\pi}\left|\int_0^{2\pi} d\varphi \exp\left(-in\varphi\right) f''\left(\varphi\right)\right| \\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi} d\varphi \left|\exp\left(-in\varphi\right) f''\left(\varphi\right)\right| \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\varphi \left|\exp\left(-in\varphi\right)\right| \left|f''\left(\varphi\right)\right| \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\varphi \left|f''\left(\varphi\right)\right|.
\end{aligned}$$

Logo, como

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi \left|f''\left(\varphi\right)\right| < \infty,$$

segue que $n^2 c_n$ é limitado quando $n \to \infty$, isto é,

$$\begin{aligned}
\lim_{n\to\infty} \left|c_n\right| &\leqslant \lim_{n\to\infty} \frac{\frac{1}{2\pi}\int_0^{2\pi} d\varphi \left|f''\left(\varphi\right)\right|}{n^2} \\
&= 0.
\end{aligned}$$

O que acontece agora com a série

$$S \equiv \sum_{n=-\infty}^{+\infty} \left|c_n\right|?$$

Acabamos de ver que

$$\left|n^2 c_n\right| \leqslant \Xi,$$

com

$$\Xi \equiv \frac{1}{2\pi}\int_0^{2\pi} d\varphi \left|f''\left(\varphi\right)\right|.$$

Portanto,

$$\left|c_n\right| \leqslant \frac{\Xi}{n^2}$$

e, assim,

$$\begin{aligned}\sum_{n=1}^{+\infty}|c_n| &\leqslant \sum_{n=1}^{+\infty}\frac{\Xi}{n^2}\\ &= \Xi\sum_{n=1}^{+\infty}\frac{1}{n^2}.\end{aligned}$$

$$a_n = \frac{1}{n^2}$$

$$a_{n+1} = \frac{1}{(n+1)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2}$$

Há a chamada função zeta de Riemann,

$$\zeta(s) \equiv \sum_{n=1}^{+\infty}\frac{1}{n^s}$$

e temos um resultado, que poderemos ver no fim deste curso, que dá

$$\zeta(2) = \frac{\pi^2}{6}.$$

Então,

$$\sum_{n=1}^{+\infty}|c_n| \leqslant \Xi\frac{\pi^2}{6}.$$

Mas, também,

$$\begin{aligned}\sum_{n=-\infty}^{-1}|c_n| &\leqslant \sum_{n=-\infty}^{-1}\frac{\Xi}{n^2}\\ &= \Xi\sum_{n=1}^{+\infty}\frac{1}{n^2}\\ &= \Xi\frac{\pi^2}{6}.\end{aligned}$$

Portanto,

$$S = \sum_{n=-\infty}^{+\infty}|c_n|$$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{-1} |c_n| + |c_0| + \sum_{n=1}^{+\infty} |c_n| \\
&\leqslant \Xi\frac{\pi^2}{6} + |c_0| + \Xi\frac{\pi^2}{6}
\end{aligned}
$$

e, assim,

$$
\sum_{n=-\infty}^{+\infty} |c_n| \quad \leqslant \quad \Xi\frac{\pi^2}{3} + |c_0| < \infty,
$$

ou seja, esta série converge.

Olhando agora a Eq. (1), vemos que

$$
\begin{aligned}
\left|\sum_{n=-\infty}^{+\infty} c_n \exp{(in\theta)}\right| &\leqslant \sum_{n=-\infty}^{+\infty} |c_n| \left|\exp{(in\theta)}\right| \\
&\leqslant \sum_{n=-\infty}^{+\infty} \Xi\frac{\pi^2}{3} + |c_0| < \infty,
\end{aligned}
$$

mostrando que a série de Fourier, neste caso, converge absolutamente para todo θ.