Agora devemos calcular uma expansão para o potencial vetorial. Analogamente ao procedimento que seguimos anteriormente, temos

$$\begin{aligned}\frac{\mathbf{J}\left(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|} &\approx \frac{1}{r}\sum_{m=0}^{\infty}\alpha_m\left(\frac{r'}{r}\right)^m\sum_{n=0}^{\infty}\frac{1}{n!}\left(\Delta t'\right)^n\frac{\partial^n}{\partial t^n}\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\\ &+ \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{rc}\sum_{m=0}^{\infty}\alpha_m\left(\frac{r'}{r}\right)^m\sum_{n=0}^{\infty}\frac{1}{n!}\left(\Delta t'\right)^n\frac{\partial^{n+1}}{\partial t^{n+1}}\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right),\end{aligned}$$

que, desprezando ordens superiores à primeira de

$$\frac{r'}{r},$$

resulta em

$$\begin{aligned}\frac{\mathbf{J}\left(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|} &\approx \frac{1}{r}\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r^2}\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\\ &+ \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{rc}\frac{\partial}{\partial t}\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right).\end{aligned}$$

Então,

$$\begin{aligned}\mathbf{A}\left(\mathbf{r},t\right) &\approx \frac{1}{rc}\int_V d^3r'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)+\frac{1}{cr^2}\int_V d^3r'\hat{\mathbf{r}}\cdot\mathbf{r}'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\\ &+ \frac{1}{rc^2}\frac{\partial}{\partial t}\int_V d^3r'\hat{\mathbf{r}}\cdot\mathbf{r}'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right).\end{aligned}$$

Consideremos a identidade vetorial

$$\hat{\mathbf{r}}\times\left[\mathbf{r}'\times\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\right] = \mathbf{r}'\hat{\mathbf{r}}\cdot\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)-\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\hat{\mathbf{r}}\cdot\mathbf{r}'.$$

Assim,

$$\begin{aligned}\int_V d^3r'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\hat{\mathbf{r}}\cdot\mathbf{r}' &= \int_V d^3r'\mathbf{r}'\hat{\mathbf{r}}\cdot\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\\ &- \hat{\mathbf{r}}\times\left[\int_V d^3r'\mathbf{r}'\times\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\right]. \qquad (3)\end{aligned}$$

Calculemos:

$$\begin{aligned}\int_V d^3r'\mathbf{r}'\hat{\mathbf{r}}\cdot\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right) &= \frac{1}{r}\int_V d^3r'\mathbf{r}'\mathbf{r}\cdot\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\\ &= \frac{1}{r}\sum_{m=1}^{3}\int_V d^3r'\mathbf{r}'x_mJ_m\left(\mathbf{r}',t-\frac{r}{c}\right)\\ &= \sum_{m=1}^{3}\frac{x_m}{r}\int_V d^3r'\mathbf{r}'J_m\left(\mathbf{r}',t-\frac{r}{c}\right)\\ &= \sum_{m=1}^{3}\frac{x_m}{r}\sum_{n=1}^{3}\hat{\mathbf{x}}_n\int_V d^3r'x'_nJ_m\left(\mathbf{r}',t-\frac{r}{c}\right). \qquad (4)\end{aligned}$$

Notemos que

$$\mathbf{r}' = \sum_{n=1}^{3} \hat{\mathbf{x}}_n x'_n$$

e

$$\boldsymbol{\nabla}' \equiv \sum_{k=1}^{3} \hat{\mathbf{x}}_k \frac{\partial}{\partial x'_k}.$$

Então, segue que

$$\begin{aligned}
\boldsymbol{\nabla}' x'_m &= \sum_{k=1}^{3} \hat{\mathbf{x}}_k \frac{\partial x'_m}{\partial x'_k} \\
&= \sum_{k=1}^{3} \hat{\mathbf{x}}_k \delta_{mk} \\
&= \hat{\mathbf{x}}_m,
\end{aligned}$$

ou seja,

$$\hat{\mathbf{x}}_m = \boldsymbol{\nabla}' x'_m.$$

Agora, consideremos

$$\begin{aligned}
\int_V d^3r' x'_n J_m \left(\mathbf{r}', t - \frac{r}{c}\right) &= \int_V d^3r' x'_n \hat{\mathbf{x}}_m \cdot \mathbf{J} \left(\mathbf{r}', t - \frac{r}{c}\right) \\
&= \int_V d^3r' x'_n \left(\boldsymbol{\nabla}' x'_m\right) \cdot \mathbf{J} \left(\mathbf{r}', t - \frac{r}{c}\right) \\
&= \int_V d^3r' \boldsymbol{\nabla}' \cdot \left[x'_n x'_m \mathbf{J} \left(\mathbf{r}', t - \frac{r}{c}\right)\right] - \int_V d^3r' x'_m \boldsymbol{\nabla}' \cdot \left[x'_n \mathbf{J} \left(\mathbf{r}', t - \frac{r}{c}\right)\right] \\
&= \oint_{S(V)} da'\, x'_m x'_n \hat{\mathbf{n}}' \cdot \mathbf{J} \left(\mathbf{r}', t - \frac{r}{c}\right) - \int_V d^3r' x'_m \boldsymbol{\nabla}' \cdot \left[x'_n \mathbf{J} \left(\mathbf{r}', t - \frac{r}{c}\right)\right] \\
&= -\int_V d^3r' x'_m \boldsymbol{\nabla}' \cdot \left[x'_n \mathbf{J} \left(\mathbf{r}', t - \frac{r}{c}\right)\right] \\
&= -\int_V d^3r' x'_m \left(\boldsymbol{\nabla}' x'_n\right) \cdot \mathbf{J} \left(\mathbf{r}', t - \frac{r}{c}\right) - \int_V d^3r' x'_m x'_n \boldsymbol{\nabla}' \cdot \mathbf{J} \left(\mathbf{r}', t - \frac{r}{c}\right) \\
&= -\int_V d^3r' x'_m \hat{\mathbf{x}}_n \cdot \mathbf{J} \left(\mathbf{r}', t - \frac{r}{c}\right) + \int_V d^3r' x'_m x'_n \frac{\partial \rho \left(\mathbf{r}', t - \frac{r}{c}\right)}{\partial t} \\
&= -\int_V d^3r' x'_m J_n \left(\mathbf{r}', t - \frac{r}{c}\right) + \frac{\partial}{\partial t} \int_V d^3r' x'_m x'_n \rho \left(\mathbf{r}', t - \frac{r}{c}\right).
\end{aligned}$$

Portanto, substituindo esse resultado na Eq. (4), obtemos

$$\begin{aligned}
\int_V d^3r'\mathbf{r}'\hat{\mathbf{r}}\cdot\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right) &= \sum_{m=1}^{3}\frac{x_m}{r}\sum_{n=1}^{3}\hat{\mathbf{x}}_n\int_V d^3r'x_n'J_m\left(\mathbf{r}',t-\frac{r}{c}\right)\\
&= -\sum_{m=1}^{3}\frac{x_m}{r}\sum_{n=1}^{3}\hat{\mathbf{x}}_n\int_V d^3r'x_m'J_n\left(\mathbf{r}',t-\frac{r}{c}\right)+\sum_{m=1}^{3}\frac{x_m}{r}\sum_{n=1}^{3}\hat{\mathbf{x}}_n\frac{\partial}{\partial t}\int_V d^3r'x_m'x_n'\rho\left(\mathbf{r}',t-\frac{r}{c}\right)\\
&= -\sum_{m=1}^{3}\frac{x_m}{r}\int_V d^3r'x_m'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)+\sum_{m=1}^{3}\frac{x_m}{r}\frac{\partial}{\partial t}\int_V d^3r'x_m'\mathbf{r}'\rho\left(\mathbf{r}',t-\frac{r}{c}\right)\\
&= -\frac{1}{r}\int_V d^3r'\mathbf{r}\cdot\mathbf{r}'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)+\frac{1}{r}\frac{\partial}{\partial t}\int_V d^3r'\mathbf{r}\cdot\mathbf{r}'\mathbf{r}'\rho\left(\mathbf{r}',t-\frac{r}{c}\right)\\
&= -\int_V d^3r'\hat{\mathbf{r}}\cdot\mathbf{r}'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)+\frac{\partial}{\partial t}\int_V d^3r'\hat{\mathbf{r}}\cdot\mathbf{r}'\mathbf{r}'\rho\left(\mathbf{r}',t-\frac{r}{c}\right).
\end{aligned}$$

Substituindo esse resultado na Eq. (3), concluímos que

$$\begin{aligned}
\int_V d^3r'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\hat{\mathbf{r}}\cdot\mathbf{r}' &= -\int_V d^3r'\hat{\mathbf{r}}\cdot\mathbf{r}'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)+\frac{\partial}{\partial t}\int_V d^3r'\hat{\mathbf{r}}\cdot\mathbf{r}'\mathbf{r}'\rho\left(\mathbf{r}',t-\frac{r}{c}\right)\\
&- \hat{\mathbf{r}}\times\left[\int_V d^3r'\mathbf{r}'\times\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\right],
\end{aligned}$$

ou seja,

$$\int_V d^3r'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\hat{\mathbf{r}}\cdot\mathbf{r}' = \left[\frac{1}{2}\int_V d^3r'\mathbf{r}'\times\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\right]\times\hat{\mathbf{r}}+\frac{1}{2}\frac{\partial}{\partial t}\int_V d^3r'\hat{\mathbf{r}}\cdot\mathbf{r}'\mathbf{r}'\rho\left(\mathbf{r}',t-\frac{r}{c}\right).$$

Definimos o momento de dipolo magnético da distribuição $\mathbf{J}$ como

$$\mathbf{m}(t) = \frac{1}{2}\int_V d^3r'\mathbf{r}'\times\mathbf{J}(\mathbf{r}',t).$$

Definimos, também, o objeto

$$\overleftrightarrow{\boldsymbol{\Upsilon}}(t) = \frac{1}{2}\int_V d^3r'\mathbf{r}'\mathbf{r}'\rho(\mathbf{r}',t),$$

que é o que se chama díade. Essa díade está relacionada com o momento de quadrupolo elétrico da distribuição. Para entendermos como é essa relação, escrevemos

$$\overleftrightarrow{\boldsymbol{\Upsilon}}\left(t-\frac{r}{c}\right) = \frac{1}{2}\sum_{m=1}^{3}\sum_{n=1}^{3}\hat{\mathbf{x}}_m\hat{\mathbf{x}}_n\int_V d^3r'x_m'x_n'\rho\left(\mathbf{r}',t-\frac{r}{c}\right),$$

onde os produtos

$$\hat{\mathbf{x}}_m\hat{\mathbf{x}}_n,\ \text{para } m,n=1,2,3,$$

são chamados produtos diádicos; notemos que não há ponto entre cada vetor do produto e os vetores são simplesmente escritos um ao lado do outro. No produto diádico, a ordem é importante, pois o produto diádico não é comutativo, já que, em geral, para qualquer vetor $\mathbf{w}$, temos

$$\mathbf{w}\cdot(\hat{\mathbf{x}}_m\hat{\mathbf{x}}_n) \neq \mathbf{w}\cdot(\hat{\mathbf{x}}_n\hat{\mathbf{x}}_m)$$

e

$$(\hat{\mathbf{x}}_m \hat{\mathbf{x}}_n) \cdot \mathbf{w} \quad \neq \quad (\hat{\mathbf{x}}_n \hat{\mathbf{x}}_m) \cdot \mathbf{w}.$$

A integral acima pode ser escrita em termos do momento de quadrupolo elétrico:

$$\begin{aligned}\int_V d^3r' x'_m x'_n \rho\left(\mathbf{r}', t - \frac{r}{c}\right) &= \frac{1}{3}\int_V d^3r' \left(3x'_m x'_n - \delta_{m,n}\left|\mathbf{r}'\right|^2\right)\rho\left(\mathbf{r}', t-\frac{r}{c}\right)\\ &+ \frac{\delta_{m,n}}{3}\int_V d^3r' \left|\mathbf{r}'\right|^2 \rho\left(\mathbf{r}', t-\frac{r}{c}\right),\end{aligned}$$

que pode ser reescrita como

$$\int_V d^3r' x'_m x'_n \rho\left(\mathbf{r}', t - \frac{r}{c}\right) = \frac{1}{3}Q_{m,n}\left(t-\frac{r}{c}\right) + \frac{\delta_{m,n}}{3} r_2\left(t-\frac{r}{c}\right),$$

onde definimos o tensor quadrupolar elétrico instantâneo como na eletrostática, isto é,

$$Q_{m,n}\left(t-\frac{r}{c}\right) = \int_V d^3r' \left(3x'_m x'_n - \delta_{m,n}\left|\mathbf{r}'\right|^2\right)\rho\left(\mathbf{r}', t-\frac{r}{c}\right)$$

e introduzimos a quantidade

$$r_2\left(t-\frac{r}{c}\right) = \int_V d^3r' \left|\mathbf{r}'\right|^2 \rho\left(\mathbf{r}', t-\frac{r}{c}\right),$$

que é igual ao traço da matriz cujos elementos são as integrais

$$\int_V d^3r' x'_m x'_n \rho\left(\mathbf{r}', t - \frac{r}{c}\right), \text{ para } m, n = 1, 2, 3.$$

Assim, podemos escrever

$$\begin{aligned}\overleftrightarrow{\boldsymbol{\Upsilon}}\left(t-\frac{r}{c}\right) &= \frac{1}{2}\sum_{m=1}^{3}\sum_{n=1}^{3}\hat{\mathbf{x}}_m\hat{\mathbf{x}}_n\left[\frac{1}{3}Q_{m,n}\left(t-\frac{r}{c}\right) + \frac{\delta_{m,n}}{3}r_2\left(t-\frac{r}{c}\right)\right]\\ &= \frac{\overleftrightarrow{Q}}{6} + \frac{\overleftrightarrow{I}}{6} r_2\left(t-\frac{r}{c}\right),\end{aligned}$$

onde

$$\overleftrightarrow{Q} = \sum_{m=1}^{3}\sum_{n=1}^{3}\hat{\mathbf{x}}_m Q_{m,n}\left(t-\frac{r}{c}\right)\hat{\mathbf{x}}_n$$

e

$$\overleftrightarrow{I} = \sum_{m=1}^{3}\hat{\mathbf{x}}_m\hat{\mathbf{x}}_m,$$

que é a identidade diádica, já que, para qualquer vetor $\mathbf{w}$, temos

$$\begin{aligned}\mathbf{w}\cdot\overleftrightarrow{I} &= \overleftrightarrow{I}\cdot\mathbf{w}\\ &= \mathbf{w}.\end{aligned}$$

Voltemos agora ao cálculo do potencial vetorial, utilizando os resultados acima. Logo,

$$\begin{aligned}
\mathbf{A}(\mathbf{r},t) &\approx \frac{1}{rc}\int_V d^3r'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)+\frac{1}{cr^2}\left[\mathbf{m}\left(t-\frac{r}{c}\right)\times\hat{\mathbf{r}}+\hat{\mathbf{r}}\cdot\frac{\partial}{\partial t}\overleftrightarrow{\boldsymbol{\Upsilon}}\left(t-\frac{r}{c}\right)\right] \\
&+ \frac{1}{rc^2}\frac{\partial}{\partial t}\left[\mathbf{m}\left(t-\frac{r}{c}\right)\times\hat{\mathbf{r}}+\hat{\mathbf{r}}\cdot\frac{\partial}{\partial t}\overleftrightarrow{\boldsymbol{\Upsilon}}\left(t-\frac{r}{c}\right)\right].
\end{aligned}$$

Para simplificar o primeiro termo, calculemos, por exemplo,

$$\begin{aligned}
\int_V d^3r'J_x\left(\mathbf{r}',t-\frac{r}{c}\right) &= \int_V d^3r'\hat{\mathbf{x}}\cdot\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right) \\
&= \int_V d^3r'\left(\boldsymbol{\nabla}'x'\right)\cdot\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right) \\
&= \int_V d^3r'\boldsymbol{\nabla}'\cdot\left[x'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\right]-\int_V d^3r'x'\boldsymbol{\nabla}'\cdot\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right).
\end{aligned}$$

Se utilizamos o teorema da divergência de Gauss, obtemos

$$\begin{aligned}
\int_V d^3r'\boldsymbol{\nabla}'\cdot\left[x'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\right] &= \oint_{S(V)} da'\hat{\mathbf{n}}'\cdot\left[x'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\right] \\
&= \oint_{S(V)} da'\,x'\hat{\mathbf{n}}'\cdot\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right) \\
&= 0,
\end{aligned}$$

pois, como as cargas e correntes estão confinadas na região V,

$$\hat{\mathbf{n}}'\cdot\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right) = 0$$

sobre a fronteira de V. Logo,

$$\int_V d^3r'J_x\left(\mathbf{r}',t-\frac{r}{c}\right) = -\int_V d^3r'x'\boldsymbol{\nabla}'\cdot\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right).$$

No entanto, utilizando a equação da continuidade, temos

$$\boldsymbol{\nabla}'\cdot\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right) = -\frac{\partial\rho\left(\mathbf{r}',t-\frac{r}{c}\right)}{\partial t}$$

e obtemos

$$\begin{aligned}
\int_V d^3r'J_x\left(\mathbf{r}',t-\frac{r}{c}\right) &= \frac{\partial}{\partial t}\int_V d^3r'x'\rho\left(\mathbf{r}',t-\frac{r}{c}\right) \\
&= \dot{p}_x\left(t-\frac{r}{c}\right).
\end{aligned}$$

Assim,

$$\begin{aligned}
\mathbf{A}(\mathbf{r},t) &\approx \frac{\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)}{rc}+\frac{1}{cr^2}\left[\mathbf{m}\left(t-\frac{r}{c}\right)\times\hat{\mathbf{r}}+\hat{\mathbf{r}}\cdot\dot{\overleftrightarrow{\boldsymbol{\Upsilon}}}\left(t-\frac{r}{c}\right)\right] \\
&+ \frac{1}{rc^2}\left[\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\times\hat{\mathbf{r}}+\hat{\mathbf{r}}\cdot\ddot{\overleftrightarrow{\boldsymbol{\Upsilon}}}\left(t-\frac{r}{c}\right)\right].
\end{aligned}$$

Para o cálculo do campo elétrico, precisamos da derivada temporal do potencial vetorial, isto é,

$$
\begin{aligned}
\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} &\approx \frac{\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)}{rc^2} \\
&+ \frac{1}{r^2c^2}\frac{\partial}{\partial t}\left[\mathbf{m}\left(t-\frac{r}{c}\right)\times\hat{\mathbf{r}}+\hat{\mathbf{r}}\cdot\dot{\overleftrightarrow{\boldsymbol{\Upsilon}}}\left(t-\frac{r}{c}\right)\right] \\
&+ \frac{1}{rc^3}\frac{\partial}{\partial t}\left[\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\times\hat{\mathbf{r}}+\hat{\mathbf{r}}\cdot\ddot{\overleftrightarrow{\boldsymbol{\Upsilon}}}\left(t-\frac{r}{c}\right)\right]
\end{aligned}
$$

e do gradiente do potencial escalar, isto é,

$$
\boldsymbol{\nabla}\phi(\mathbf{r},t) \approx \boldsymbol{\nabla}\left(\frac{Q}{r}\right)+\boldsymbol{\nabla}\left[\frac{\hat{\mathbf{r}}\cdot\mathbf{p}\left(t-\frac{r}{c}\right)}{r^2}\right]+\boldsymbol{\nabla}\left[\frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)}{rc}\right].
$$

O campo elétrico de radiação, definido como sendo apenas a parte que varia com o inverso de r, pode ser escrito como

$$
\begin{aligned}
\mathbf{E}_{\text{rad}} &\approx -\frac{1}{rc}\boldsymbol{\nabla}\left[\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right]-\frac{\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)}{rc^2} \\
&- \frac{1}{rc^3}\frac{\partial}{\partial t}\left[\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\times\hat{\mathbf{r}}+\hat{\mathbf{r}}\cdot\ddot{\overleftrightarrow{\boldsymbol{\Upsilon}}}\left(t-\frac{r}{c}\right)\right].
\end{aligned}
$$

Calculemos:

$$
\begin{aligned}
\boldsymbol{\nabla}\left[\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right] &\approx \boldsymbol{\nabla}\left[\frac{1}{r}\mathbf{r}\cdot\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right] \\
&= \left[\boldsymbol{\nabla}\frac{1}{r}\right]\mathbf{r}\cdot\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)+\frac{1}{r}\boldsymbol{\nabla}\left[\mathbf{r}\cdot\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right] \\
&= -\frac{\hat{\mathbf{r}}}{r^2}\mathbf{r}\cdot\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)+\frac{1}{r}\boldsymbol{\nabla}\left[\sum_{i=1}^{3}x_i\dot{p}_i\left(t-\frac{r}{c}\right)\right] \\
&= -\frac{\hat{\mathbf{r}}}{r^2}\mathbf{r}\cdot\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)+\frac{1}{r}\sum_{i=1}^{3}\left(\boldsymbol{\nabla}x_i\right)\dot{p}_i\left(t-\frac{r}{c}\right)+\frac{1}{r}\sum_{i=1}^{3}x_i\left[\boldsymbol{\nabla}\dot{p}_i\left(t-\frac{r}{c}\right)\right] \\
&= -\frac{\hat{\mathbf{r}}}{r^2}\mathbf{r}\cdot\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)+\frac{1}{r}\sum_{i=1}^{3}\hat{\mathbf{x}}_i\dot{p}_i\left(t-\frac{r}{c}\right)-\frac{1}{rc}\sum_{i=1}^{3}x_i\ddot{p}_i\left(t-\frac{r}{c}\right)\left[\boldsymbol{\nabla}r\right] \\
&= -\frac{\hat{\mathbf{r}}}{r^2}\mathbf{r}\cdot\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)+\frac{1}{r}\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)-\frac{1}{rc}\mathbf{r}\cdot\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\hat{\mathbf{r}}.
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
\mathbf{E}_{\text{rad}} &\approx \frac{\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)-\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)}{rc^2}-\frac{1}{rc^3}\frac{\partial}{\partial t}\left[\dot{\mathbf{m}}\left(t-\frac{r}{c}\right)\times\hat{\mathbf{r}}+\hat{\mathbf{r}}\cdot\ddot{\overleftrightarrow{\boldsymbol{\Upsilon}}}\left(t-\frac{r}{c}\right)\right] \\
&= \frac{d^2}{dt^2}\left[\frac{\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{p}\left(t-\frac{r}{c}\right)-\mathbf{p}\left(t-\frac{r}{c}\right)}{rc^2}-\frac{\mathbf{m}\left(t-\frac{r}{c}\right)\times\hat{\mathbf{r}}+\hat{\mathbf{r}}\cdot\dot{\overleftrightarrow{\boldsymbol{\Upsilon}}}\left(t-\frac{r}{c}\right)}{rc^3}\right] \\
&= \frac{1}{4\pi\varepsilon_0 rc^2}\frac{d^2}{dt^2}\left[\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{p}\left(t-\frac{r}{c}\right)-\mathbf{p}\left(t-\frac{r}{c}\right)+\frac{\hat{\mathbf{r}}\times\mathbf{m}\left(t-\frac{r}{c}\right)-\hat{\mathbf{r}}\cdot\dot{\overleftrightarrow{\boldsymbol{\Upsilon}}}\left(t-\frac{r}{c}\right)}{c}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4\pi\varepsilon_0 r c^2}\frac{d^2}{dt^2}\left\{\hat{\mathbf{r}}\times\left[\hat{\mathbf{r}}\times\mathbf{p}\left(t-\frac{r}{c}\right)\right]+\frac{\hat{\mathbf{r}}\times\mathbf{m}\left(t-\frac{r}{c}\right)-\hat{\mathbf{r}}\cdot\dot{\overleftrightarrow{\boldsymbol{\Upsilon}}}\left(t-\frac{r}{c}\right)}{c}\right\} \\
&= \frac{1}{4\pi\varepsilon_0 r c^2}\frac{d^2}{dt^2}\left\{\hat{\mathbf{r}}\times\left[\hat{\mathbf{r}}\times\mathbf{p}\left(t-\frac{r}{c}\right)+\frac{\mathbf{m}\left(t-\frac{r}{c}\right)}{c}\right]-\frac{\hat{\mathbf{r}}\cdot\dot{\overleftrightarrow{\boldsymbol{\Upsilon}}}\left(t-\frac{r}{c}\right)}{c}\right\}.
\end{aligned}
$$

onde já desprezamos todos os termos que variam com o inverso de r^2. Comparemos

$$
\mathbf{p}\left(t-\frac{r}{c}\right) = \int_V d^3r'\mathbf{r}'\rho\left(\mathbf{r}',t-\frac{r}{c}\right)
$$

e

$$
\mathbf{m}\left(t-\frac{r}{c}\right) = \frac{1}{2}\int_V d^3r'\mathbf{r}'\times\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right).
$$

Mas,

$$
\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right) = \mathbf{v}\left(\mathbf{r}',t-\frac{r}{c}\right)\rho\left(\mathbf{r}',t-\frac{r}{c}\right),
$$

onde $\mathbf{v}\left(\mathbf{r}',t-\frac{r}{c}\right)$ é o campo de velocidades das partículas carregadas na região V. Assim, comparativamente, o momento de dipolo magnético dividido por c é uma ordem v/c menor do que o momento de dipólo elétrico. Analogamente, comparemos

$$
\mathbf{p}\left(t-\frac{r}{c}\right) = \int_V d^3r'\mathbf{r}'\rho\left(\mathbf{r}',t-\frac{r}{c}\right)
$$

e

$$
\begin{aligned}
\hat{\mathbf{r}}\cdot\dot{\overleftrightarrow{\boldsymbol{\Upsilon}}}\left(t-\frac{r}{c}\right) &= \frac{1}{2}\int_V d^3r'\hat{\mathbf{r}}\cdot\mathbf{r}'\mathbf{r}'\frac{\partial\rho\left(\mathbf{r}',t-\frac{r}{c}\right)}{\partial t} \\
&= -\frac{1}{2}\sum_{n=1}^{3}\hat{\mathbf{x}}_n\int_V d^3r'\hat{\mathbf{r}}\cdot\mathbf{r}'x_n'\boldsymbol{\nabla}'\cdot\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right) \\
&= -\frac{1}{2}\sum_{n=1}^{3}\hat{\mathbf{x}}_n\int_V d^3r'\boldsymbol{\nabla}'\cdot\left[\hat{\mathbf{r}}\cdot\mathbf{r}'x_n'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\right]+\frac{1}{2}\sum_{n=1}^{3}\hat{\mathbf{x}}_n\int_V d^3r'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\cdot\boldsymbol{\nabla}'\left(\hat{\mathbf{r}}\cdot\mathbf{r}'x_n'\right).
\end{aligned}
$$

Mas,

$$
\begin{aligned}
\int_V d^3r'\boldsymbol{\nabla}'\cdot\left[\hat{\mathbf{r}}\cdot\mathbf{r}'x_n'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\right] &= \oint_{S(V)} da'\hat{\mathbf{n}}'\cdot\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\hat{\mathbf{r}}\cdot\mathbf{r}'x_n' \\
&= 0
\end{aligned}
$$

e

$$
\begin{aligned}
\boldsymbol{\nabla}'\left(\hat{\mathbf{r}}\cdot\mathbf{r}'x_n'\right) &= \sum_{m=1}^{3}\frac{x_m}{r}\boldsymbol{\nabla}'\left(x_m'x_n'\right) \\
&= \sum_{m=1}^{3}\frac{x_m}{r}\left(\hat{\mathbf{x}}_m x_n'+\hat{\mathbf{x}}_n x_m'\right) \\
&= \hat{\mathbf{r}}x_n'+\hat{\mathbf{x}}_n\hat{\mathbf{r}}\cdot\mathbf{r}'.
\end{aligned}
$$

Então,

$$\begin{aligned}
\hat{\mathbf{r}}\cdot\dot{\overleftrightarrow{\boldsymbol{\Upsilon}}}\left(t-\frac{r}{c}\right) &= \frac{1}{2}\sum_{n=1}^{3}\hat{\mathbf{x}}_n\int_V d^3r'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\cdot\left(\hat{\mathbf{r}}x_n'+\hat{\mathbf{x}}_n\hat{\mathbf{r}}\cdot\mathbf{r}'\right)\\
&= \frac{1}{2}\int_V d^3r'\left[\mathbf{r}'\hat{\mathbf{r}}\cdot\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)+\hat{\mathbf{r}}\cdot\mathbf{r}'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)\right]\\
&= \frac{1}{2}\int_V d^3r'\left[\mathbf{r}'\hat{\mathbf{r}}\cdot\mathbf{v}\left(\mathbf{r}',t-\frac{r}{c}\right)+\hat{\mathbf{r}}\cdot\mathbf{r}'\mathbf{v}\left(\mathbf{r}',t-\frac{r}{c}\right)\right]\rho\left(\mathbf{r}',t-\frac{r}{c}\right).
\end{aligned}$$

Assim, comparativamente, o termo

$$\hat{\mathbf{r}}\cdot\dot{\overleftrightarrow{\boldsymbol{\Upsilon}}}\left(t-\frac{r}{c}\right)$$

dividido por c é uma ordem v/c menor do que o momento de dipolo elétrico. Podemos, portanto, desprezar os termos envolvendo $\mathbf{m}$ e

$$\dot{\overleftrightarrow{\boldsymbol{\Upsilon}}}$$

e o campo elétrico de radiação fica

$$\mathbf{E}_{\mathrm{rad}} \approx \frac{\hat{\mathbf{r}}\times\left[\hat{\mathbf{r}}\times\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right]}{rc^2}.$$

Para sermos consistentes, devemos adotar os potenciais de radiação:

$$\phi_{\mathrm{rad}}(\mathbf{r},t) \approx \frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)}{rc}$$

e

$$\mathbf{A}_{\mathrm{rad}}(\mathbf{r},t) \approx \frac{\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)}{cr},$$

já que os outros termos não contribuem para os campos de radiação, que, por definição, devem variar com o inverso de r. Logo,

$$\begin{aligned}
\mathbf{B}_{\mathrm{rad}}(\mathbf{r},t) &\approx \boldsymbol{\nabla}\times\mathbf{A}_{\mathrm{rad}}(\mathbf{r},t)\\
&\approx \frac{\mu_0}{4\pi r}\boldsymbol{\nabla}\times\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)\\
&= \frac{\mu_0}{4\pi rc}\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\times\boldsymbol{\nabla}r\\
&= \frac{\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\times\hat{\boldsymbol{r}}}{rc^2}.
\end{aligned}$$

Resumindo os resultados até este ponto, o campo elétrico de radiação que calculamos deu

$$\mathbf{E}_{\mathrm{rad}} \approx \frac{\hat{\mathbf{r}}\times\left[\hat{\mathbf{r}}\times\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\right]}{rc^2}.$$

Para sermos consistentes, devemos adotar os potenciais de radiação:

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$$\mathbf{A}_{\text{rad}}(\mathbf{r},t) \approx \frac{\dot{\mathbf{p}}\left(t-\frac{r}{c}\right)}{rc},$$

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&= \frac{\mu_0}{4\pi rc}\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\times\boldsymbol{\nabla}r \\
&= \frac{\ddot{\mathbf{p}}\left(t-\frac{r}{c}\right)\times\hat{\boldsymbol{r}}}{rc^2}.
\end{aligned}$$

Notemos que, em todas as passagens para calcular os campos e potenciais de radiação, formalmente estamos fazendo o limite, por exemplo,

$$\mathbf{B}_{\text{rad}}(\mathbf{r},t) \equiv \lim_{r\to\infty}\left[r\mathbf{B}_{\text{total}}(\mathbf{r},t)\right].$$