Calculando o Campo Indução Magnética e o Campo Elétrico

Vamos agora calcular os campos $\mathbf{B}$ e $\mathbf{E}$. Comecemos com o cálculo de $\mathbf{B}$:

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

ou, em termos de componentes,

$$B_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} \partial_j A_k,$$

onde ε_{ijk} é o símbolo de Levi-Civita, que é dado por

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{se } (i,j,k) \text{ for uma permutação par de } (1,2,3), \\ 0, & \text{se pelo menos dois dos índices } i,j,k \text{ forem iguais e} \\ -1, & \text{se } (i,j,k) \text{ for uma permutação ímpar de } (1,2,3). \end{cases}$$

Também utilizemos a notação

$$\partial_j = \frac{\partial}{\partial x_j},$$

para $j = 1, 2, 3$. A convenção de Einstein para somas permite que escrevamos

$$B_i = \varepsilon_{ijk} \partial_j A_k,$$

onde subentendemos que os índices j e k estão somados de 1 a 3, porque aparecem repetidos no mesmo termo. Temos, assim,

$$\begin{aligned} \partial_j A_k &= \frac{q}{c} \partial_j \left(\frac{v_k}{R - \mathbf{R} \cdot \boldsymbol{\beta}} \right) \\ &= \frac{q}{c} \left[\frac{\partial_j v_k}{R - \mathbf{R} \cdot \boldsymbol{\beta}} + v_k \partial_j \left(\frac{1}{R - \mathbf{R} \cdot \boldsymbol{\beta}} \right) \right] \\ &= \frac{q}{c} \left[\frac{\partial_j v_k}{R - \mathbf{R} \cdot \boldsymbol{\beta}} - \frac{v_k}{(R - \mathbf{R} \cdot \boldsymbol{\beta})^2} \left(\partial_j R - \boldsymbol{\beta} \cdot \partial_j \mathbf{R} - \mathbf{R} \cdot \partial_j \boldsymbol{\beta} \right) \right]. \end{aligned}$$

Também,

$$\begin{aligned} \partial_j v_k &= \frac{dv_k}{dt_R} \partial_j t_R \\ &= a_k \partial_j t_R, \end{aligned}$$

onde

$$\begin{aligned} \mathbf{a} &\equiv \frac{d\mathbf{v}}{dt_R} \\ &= \frac{d\mathbf{v}(t_R)}{dt_R} \end{aligned}$$

e, portanto,

$$a_k = \hat{\mathbf{x}}_k \cdot \mathbf{a}.$$

Façamos agora o cálculo de $\partial_j t_R$:

$$\begin{aligned}\partial_j t_R &= \partial_j \left(t - \frac{R}{c}\right) \\ &= -\frac{1}{c}\partial_j R.\end{aligned}$$

Mas,

$$\begin{aligned}\partial_j R &= \frac{1}{2R}\partial_j R^2 \\ &= \frac{1}{2R}\partial_j \left(\mathbf{R} \cdot \mathbf{R}\right) \\ &= \frac{\mathbf{R}}{R} \cdot \partial_j \mathbf{R}\end{aligned}$$

e, portanto,

$$\begin{aligned}\partial_j t_R &= -\frac{1}{c}\frac{\mathbf{R}}{R} \cdot \partial_j \mathbf{R} \\ &= -\frac{1}{c}\hat{\mathbf{R}} \cdot \partial_j \mathbf{R}\end{aligned}$$

Como

$$\mathbf{R} = \mathbf{r} - \mathbf{r}_0\left(t_R\right),$$

segue que

$$\begin{aligned}\partial_j \mathbf{R} &= \hat{\mathbf{x}}_j - \frac{d\mathbf{r}_0\left(t_R\right)}{dt_R}\partial_j t_R \\ &= \hat{\mathbf{x}}_j - \mathbf{v}\partial_j t_R.\end{aligned}$$

Sendo assim,

$$\begin{aligned}\partial_j t_R &= -\frac{1}{c}\hat{\mathbf{R}} \cdot \partial_j \mathbf{R} \\ &= -\frac{1}{c}\hat{\mathbf{R}} \cdot \left(\hat{\mathbf{x}}_j - \mathbf{v}\partial_j t_R\right) \\ &= -\frac{1}{c}\hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j + \hat{\mathbf{R}} \cdot \boldsymbol{\beta}\partial_j t_R,\end{aligned}$$

ou seja,

$$\left(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta}\right)\partial_j t_R = -\frac{1}{c}\hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j,$$

resultando em

$$\partial_j t_R = -\frac{1}{c}\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}.$$

Portanto,

$$\partial_j v_k = -\frac{a_k}{c}\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}.$$

O próximo passo é calcularmos $\partial_j R$, que, como vimos logo acima,

$$\partial_j R = \frac{\mathbf{R}}{R}\cdot\partial_j\mathbf{R}$$

e, então, como já temos $\partial_j\mathbf{R}$, vem[1]:

$$\begin{aligned}
\partial_j R &= \hat{\mathbf{R}}\cdot(\hat{\mathbf{x}}_j - \mathbf{v}\partial_j t_R)\\
&= \hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j - \hat{\mathbf{R}}\cdot\mathbf{v}\partial_j t_R\\
&= \hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j + \frac{\hat{\mathbf{R}}\cdot\frac{\mathbf{v}}{c}}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j\\
&= \left(1+\frac{\hat{\mathbf{R}}\cdot\boldsymbol{\beta}}{1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}}\right)\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j\\
&= \frac{1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}+\hat{\mathbf{R}}\cdot\boldsymbol{\beta}}{1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}}\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j\\
&= \frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}}.
\end{aligned}$$

A seguir, o próximo termo no cálculo de $\partial_j A_k$ tem a quantidade $\partial_j\mathbf{R}$ que, já vimos que é dada por:

$$\partial_j\mathbf{R} = \hat{\mathbf{x}}_j - \mathbf{v}\partial_j t_R.$$

Logo, usando nosso resultado para $\partial_j t_R$, dá

$$\partial_j\mathbf{R} = \hat{\mathbf{x}}_j + \boldsymbol{\beta}\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}.$$

[1]Notemos que, só para esclarecer a notação, escrevemos, por exemplo,

$$\mathbf{r} = \hat{\mathbf{x}}_k x_k.$$

Logo,

$$\begin{aligned}
\mathbf{r}\cdot\hat{\mathbf{x}}_j &= (\hat{\mathbf{x}}_k\cdot\hat{\mathbf{x}}_j)\,x_k\\
&= \delta_{kj}x_k\\
&= x_j.
\end{aligned}$$

Então, segue que

$$\begin{aligned}
\boldsymbol{\beta}\cdot\partial_j\mathbf{R} &= \boldsymbol{\beta}\cdot\hat{\mathbf{x}}_j+\boldsymbol{\beta}\cdot\boldsymbol{\beta}\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)} \\
&= \boldsymbol{\beta}\cdot\hat{\mathbf{x}}_j+\beta^2\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}.
\end{aligned}$$

Finalmente, só falta calcularmos o seguinte fator:

$$\begin{aligned}
\partial_j\boldsymbol{\beta} &= \frac{1}{c}\partial_j\mathbf{v} \\
&= \frac{1}{c}\frac{d\mathbf{v}}{dt_R}\partial_j t_R \\
&= -\frac{\mathbf{a}}{c^2}\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}.
\end{aligned}$$

Assim,

$$\begin{aligned}
\partial_j A_k &= \frac{q}{c}\left[\frac{\partial_j v_k}{R-\mathbf{R}\cdot\boldsymbol{\beta}}-\frac{v_k}{\left(R-\mathbf{R}\cdot\boldsymbol{\beta}\right)^2}\left(\partial_j R-\boldsymbol{\beta}\cdot\partial_j\mathbf{R}-\mathbf{R}\cdot\partial_j\boldsymbol{\beta}\right)\right] \\
&= \frac{q}{c}\left[\frac{\partial_j v_k}{R\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}-\frac{v_k\left(\partial_j R-\boldsymbol{\beta}\cdot\partial_j\mathbf{R}-R\hat{\mathbf{R}}\cdot\partial_j\boldsymbol{\beta}\right)}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2}\right] \\
&= \frac{q}{c}\left[\frac{\partial_j v_k}{R\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}+\frac{v_k\hat{\mathbf{R}}\cdot\partial_j\boldsymbol{\beta}}{R\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2}-\frac{v_k\left(\partial_j R-\boldsymbol{\beta}\cdot\partial_j\mathbf{R}\right)}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2}\right].
\end{aligned}$$

Substituindo as derivadas parciais, temos

$$\begin{aligned}
\partial_j A_k &= \frac{q}{c}\left[\frac{-a_k\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{Rc\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2}-\frac{v_k\hat{\mathbf{R}}\cdot\mathbf{a}\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{Rc^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}\right. \\
&- \left.\frac{v_k}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2}\left(\frac{\left(1-\beta^2\right)\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}-\beta_j\right)\right]
\end{aligned}$$

onde

$$\beta_j = \frac{v_j}{c}.$$

Notemos que

$$
\begin{aligned}
\varepsilon_{ijk} v_k \beta_j &= \varepsilon_{ijk} \beta_j v_k \\
&= \frac{1}{c} \varepsilon_{ijk} v_j v_k \\
&= 0,
\end{aligned}
$$

pois, das propriedades do símbolo de Levi-Civita, vemos que

$$
\begin{aligned}
\varepsilon_{ijk} v_j v_k &= -\varepsilon_{ikj} v_j v_k \\
&= -\varepsilon_{ikj} v_k v_j
\end{aligned}
$$

e, como neste resultado, j e k são índices mudos, fazemos $j \to k$ e $k \to j$. Com isso, vem:

$$
\begin{aligned}
\varepsilon_{ijk} v_j v_k &= -\varepsilon_{ikj} v_k v_j \\
&= -\varepsilon_{ijk} v_j v_k,
\end{aligned}
$$

isto é,

$$
2\varepsilon_{ijk} v_j v_k = 0.
$$

Em termos vetoriais, podemos escrever

$$
\begin{aligned}
\mathbf{B} = \frac{q}{c} \Bigg[& \frac{-\hat{\mathbf{R}} \times \mathbf{a}}{Rc\left(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta}\right)^2} - \frac{\left(\hat{\mathbf{R}} \cdot \mathbf{a}\right) \hat{\mathbf{R}} \times \mathbf{v}}{Rc^2\left(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta}\right)^3} \\
& - \frac{\left(1 - \beta^2\right) \hat{\mathbf{R}} \times \mathbf{v}}{R^2\left(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta}\right)^3} \Bigg].
\end{aligned}
$$

Notemos que o termo proporcional a $\boldsymbol{\beta} \times \mathbf{v}$ se anula.

Post Scriptum

No final da aula, eu fotografei (veja figura)

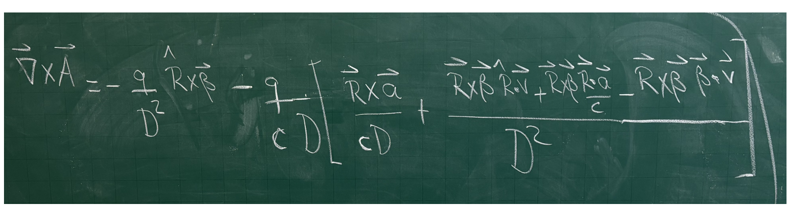

Figure 1: Fotografia da lousa no final da aula.

o que tinha na lousa, que era

$$
\boldsymbol{\nabla} \times \mathbf{A} = -\frac{q}{D^2} \hat{\mathbf{R}} \times \boldsymbol{\beta} - \frac{q}{cD} \left[\frac{\mathbf{R} \times \mathbf{a}}{cD} + \frac{\mathbf{R} \times \boldsymbol{\beta} \hat{\mathbf{R}} \cdot \mathbf{v} + \mathbf{R} \times \boldsymbol{\beta} \frac{\mathbf{R} \cdot \mathbf{a}}{c} - \mathbf{R} \times \boldsymbol{\beta} \boldsymbol{\beta} \cdot \mathbf{v}}{D^2} \right],
$$

onde

$$D = R - \mathbf{R}\cdot\boldsymbol{\beta}.$$

O primeiro termo entre colchetes pode ser separado dos demais e o termo fora dos colchetes pode ser colocado dentro. Assim,

$$\nabla\times\mathbf{A} = -\frac{q}{c}\left(\frac{\mathbf{R}\times\mathbf{a}}{cD^2}\right) - \frac{q}{cD}\left[\frac{c\hat{\mathbf{R}}\times\boldsymbol{\beta}}{D} + \frac{\mathbf{R}\times\boldsymbol{\beta}\hat{\mathbf{R}}\cdot\mathbf{v} + \mathbf{R}\times\boldsymbol{\beta}\frac{\mathbf{R}\cdot\mathbf{a}}{c} - \mathbf{R}\times\boldsymbol{\beta}\boldsymbol{\beta}\cdot\mathbf{v}}{D^2}\right].$$

Vamos trabalhar com os termos entre colchetes. Então,

$$\begin{aligned}
\nabla\times\mathbf{A} &= -\frac{q}{c}\left(\frac{\mathbf{R}\times\mathbf{a}}{cD^2}\right) - \frac{q}{cD}\left[\frac{c\hat{\mathbf{R}}\times\boldsymbol{\beta}D + \mathbf{R}\times\boldsymbol{\beta}\hat{\mathbf{R}}\cdot\mathbf{v} + \mathbf{R}\times\boldsymbol{\beta}\frac{\mathbf{R}\cdot\mathbf{a}}{c} - \mathbf{R}\times\boldsymbol{\beta}\boldsymbol{\beta}\cdot\mathbf{v}}{D^2}\right]\\
&= -\frac{q}{c}\left(\frac{\mathbf{R}\times\mathbf{a}}{cD^2}\right) - \frac{q}{cD}\left[\frac{c\hat{\mathbf{R}}\times\boldsymbol{\beta}\left(R-\mathbf{R}\cdot\boldsymbol{\beta}\right) + \mathbf{R}\times\boldsymbol{\beta}\hat{\mathbf{R}}\cdot\mathbf{v} + \mathbf{R}\times\boldsymbol{\beta}\frac{\mathbf{R}\cdot\mathbf{a}}{c} - \mathbf{R}\times\boldsymbol{\beta}\boldsymbol{\beta}\cdot\mathbf{v}}{D^2}\right]\\
&= -\frac{q}{c}\left(\frac{\mathbf{R}\times\mathbf{a}}{cD^2}\right) - \frac{q}{cD}\left[\frac{c\hat{\mathbf{R}}\times\boldsymbol{\beta}R - c\hat{\mathbf{R}}\times\boldsymbol{\beta}\mathbf{R}\cdot\boldsymbol{\beta} + \mathbf{R}\times\boldsymbol{\beta}\hat{\mathbf{R}}\cdot\mathbf{v} + \mathbf{R}\times\boldsymbol{\beta}\frac{\mathbf{R}\cdot\mathbf{a}}{c} - \mathbf{R}\times\boldsymbol{\beta}\boldsymbol{\beta}\cdot\mathbf{v}}{D^2}\right]\\
&= -\frac{q}{c}\left(\frac{\mathbf{R}\times\mathbf{a}}{cD^2}\right) - \frac{q}{cD}\left[\frac{\mathbf{R}\times\mathbf{v} - \mathbf{R}\times\boldsymbol{\beta}\hat{\mathbf{R}}\cdot\mathbf{v} + \mathbf{R}\times\boldsymbol{\beta}\hat{\mathbf{R}}\cdot\mathbf{v} + \mathbf{R}\times\boldsymbol{\beta}\frac{\mathbf{R}\cdot\mathbf{a}}{c} - \mathbf{R}\times\boldsymbol{\beta}\boldsymbol{\beta}\cdot\mathbf{v}}{D^2}\right]\\
&= -\frac{q}{c}\left(\frac{\mathbf{R}\times\mathbf{a}}{cD^2}\right) - \frac{q}{cD}\left[\frac{\mathbf{R}\times\mathbf{v} + \mathbf{R}\times\mathbf{v}\mathbf{R}\cdot\mathbf{a}/c^2 - \mathbf{R}\times\boldsymbol{\beta}\boldsymbol{\beta}\cdot\mathbf{v}}{D^2}\right]\\
&= -\frac{q}{c}\left(\frac{\mathbf{R}\times\mathbf{a}}{cD^2}\right) - \frac{q}{cD}\left[\frac{\mathbf{R}\times\mathbf{v} + \mathbf{R}\times\mathbf{v}\mathbf{R}\cdot\mathbf{a}/c^2 - \mathbf{R}\times\mathbf{v}\boldsymbol{\beta}\cdot\boldsymbol{\beta}}{D^2}\right]\\
&= -\frac{q}{c}\left(\frac{\mathbf{R}\times\mathbf{a}}{cD^2}\right) - \frac{q}{cD}\mathbf{R}\times\mathbf{v}\left(\frac{1+\mathbf{R}\cdot\mathbf{a}/c^2-\beta^2}{D^2}\right).
\end{aligned}$$

Agora podemos ver que, de fato, esse resultado ainda pode ficar igual ao encontrado acima, pois

$$\begin{aligned}
\nabla\times\mathbf{A} &= -\frac{q}{c}\left[\frac{\mathbf{R}\times\mathbf{a}}{cD^2} + \frac{\mathbf{R}\times\mathbf{v}\mathbf{R}\cdot\mathbf{a}}{c^2D^3} + \frac{\mathbf{R}\times\mathbf{v}\left(1-\beta^2\right)}{D^3}\right]\\
&= -\frac{q}{c}\left[\frac{\mathbf{R}\times\mathbf{a}}{c\left(R-\mathbf{R}\cdot\boldsymbol{\beta}\right)^2} + \frac{\mathbf{R}\times\mathbf{v}\mathbf{R}\cdot\mathbf{a}}{c^2\left(R-\mathbf{R}\cdot\boldsymbol{\beta}\right)^3} + \frac{\mathbf{R}\times\mathbf{v}\left(1-\beta^2\right)}{\left(R-\mathbf{R}\cdot\boldsymbol{\beta}\right)^3}\right]\\
&= -\frac{q}{c}\left[\frac{\hat{\mathbf{R}}\times\mathbf{a}}{cR\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2} + \frac{\hat{\mathbf{R}}\times\mathbf{v}\hat{\mathbf{R}}\cdot\mathbf{a}}{Rc^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3} + \frac{\hat{\mathbf{R}}\times\mathbf{v}\left(1-\beta^2\right)}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}\right],
\end{aligned}$$

que dá

$$\mathbf{B} = \frac{q}{c}\left[-\frac{\hat{\mathbf{R}}\times\mathbf{a}}{cR\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2} - \frac{\hat{\mathbf{R}}\times\mathbf{v}\hat{\mathbf{R}}\cdot\mathbf{a}}{Rc^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3} - \frac{\hat{\mathbf{R}}\times\mathbf{v}\left(1-\beta^2\right)}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}\right],$$

exatamente o resultado obtido acima, que é

$$\mathbf{B} = \frac{q}{c}\left[\frac{-\hat{\mathbf{R}}\times\mathbf{a}}{Rc\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2} - \frac{\left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)\hat{\mathbf{R}}\times\mathbf{v}}{Rc^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3} - \frac{\left(1-\beta^2\right)\hat{\mathbf{R}}\times\mathbf{v}}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}\right].$$

Fim do Post Scriptum. ■