and after elimination of the time t in favor of polar angle θ:

$$\frac{d^2u}{d\theta^2}+u=\frac{km}{L^2}-\frac{mB(u)}{L^2u^2}. \quad (33)$$

Here we have assumed a central force of the form (14). If the small correction B were actually zero, we immediately find the standard elliptical orbit

$$u=1/r=(1/r_0)(1-\epsilon\cos\theta), \quad (34)$$

where $r_0=L^2/km$ is the average radius.

The approach of Krylov and Bogoliubov is to seek a solution when B is nonzero of the form

$$u=(1/r_0)[1-\epsilon(\theta)\cos\phi(\theta)]. \quad (35)$$

This introduces two unknown functions, ϵ and ϕ, so Eq. (33) alone will be insufficient to determine them. As the needed second condition we ask that the derivative of u with respect to θ have the same form as when B is zero:

$$\frac{du}{d\theta}=\frac{\epsilon}{r_0}\sin\phi. \quad (36)$$

On substituting (36) into (33), and (35) into (36) we can solve for the derivatives of ϵ and ϕ with respect to θ:

$$\frac{d\epsilon}{d\theta}=-\frac{B(u)}{ku^2}\sin\phi, \quad (37)$$

$$\frac{d\phi}{d\theta}=1-\frac{B(u)}{\epsilon ku^2}\cos\phi. \quad (38)$$

The method of averages consists in approximating the right-hand sides of (37) and (38) by their averages over one period in ϕ.

To implement this, we need a further approximation for the factor B/u^2:

$$1/u^2=r_0^2/(1-\epsilon\cos\phi)^2\approx r_0^2(1+2\epsilon\cos\phi), \quad (39)$$

$$B(u)\approx B(r_0)+(r-r_0)B'(r_0)$$
$$\approx B(r_0)+\epsilon r_0\cos\phi B'(r_0), \quad (40)$$

$$B(u)/u^2\approx r_0^2\{B(r_0)$$
$$+\epsilon[2B(r_0)+r_0B'(r_0)]\cos\phi\}. \quad (41)$$

From (41) and (37) we see that ϵ is constant on average over many orbits, as expected. On combining (41) and (38) and averaging $\cos^2\phi$ to $\frac{1}{2}$, we find

$$\frac{d\phi}{d\theta}\approx 1-\frac{r_0^2}{2k}[2B(r_0)+r_0B'(r_0)]. \quad (42)$$

In the notation of Eqs. (1) and (10), we have $\phi=\alpha t$ and $\theta\approx\Omega t$, so $d\phi/d\theta\approx\alpha/\Omega$, and the angular velocity of precession is

$$\omega=\Omega-\alpha\approx(r_0^2\Omega/2k)[2B(r_0)+r_0B'(r_0)]. \quad (43)$$

Noting that $k\approx m\Omega^2r_0^3$, Eq. (43) reduces to our previous result (15).

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Faraday rotation in the undergraduate advanced laboratory

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A Faraday rotation experiment is described for laser beams of two wavelengths in moderate magnetic fields, using flint glasses of large rotary birefringence and (liquid) carbon disulfide. Results show a linear relation between angle of rotation and field strength, and indicate a strong dependence on wavelength, attributable to the dispersion, which is determined from a measurement of the Verdet constant. The theoretical treatment and experimental technique are recommended for the advanced undergraduate laboratory.

I. INTRODUCTION

Faraday rotation provides an excellent experiment for the upper division, undergraduate physics, or engineering student, combining elements of polarization optics with magnetism and atomic physics. The experiment has been made more elegant in recent times due to the availability of relatively inexpensive laser sources and rather esoteric glasses of the flint variety that show large circular birefringence. Tutorial articles[1–5] that discuss various theoretical aspects or applications of Faraday rotation have appeared in this Journal from time to time, but we are aware of only one[6] that describes an experimental study whose primary focus is the Faraday rotation itself.

The rotation of the polarization vector of light propagating within a transparent material along the direction of an external magnetic field was first reported by Michael Faraday in 1845.[7] Theoretical treatments of the magnitude of the rotation can be found in many optics textbooks or laboratory manuals.[8–10] Although a rigorous treatment requires quantum mechanics, the following is suggested as a simpler, classical treatment appropriate for the undergraduate student.

II. THE FARADAY EFFECT

When a transparent solid or liquid is placed in a magnetic field and linearly polarized light is passed through it along the direction of the magnetic field, the emerging light is found to remain linearly polarized, but with a net rotation β of the plane of polarization that is proportional both to the thickness d of the sample and the strength of the magnetic field B, i.e.,

$$\beta = VBd, \tag{1}$$

where V is the *Verdet constant* for the material, usually expressed in minutes of angle per G cm. The Verdet constant is found to depend also on the wavelength of the light. An interesting aspect of the Faraday rotation is that the sense of rotation relative to the magnetic field direction is independent of the direction of the light for a given material.[1] Equation (1) can be understood and a theoretical expression for the Verdet constant can be found in the following way.

A. The Larmor frequency

Consider the effect of an applied field on a current loop formed by the circulation of an electron. The situation is pictured in Fig. 1. An electron moves in a circle of radius r at some arbitrary orientation θ relative to the applied magnetic field B. Its angular momentum L and opposing magnetic moment μ_e are shown. Their directions are opposite because the electron has a negative charge. The magnetic field exerts a torque on the magnetic dipole represented by the circulating electron according to

$$\boldsymbol{\tau} = \boldsymbol{\mu}_e \times \mathbf{B} \quad \text{(directed into page)}.$$

The angular impulse $\tau\, dt$ produces a change in angular momentum as required by Newton's second law:

$$\boldsymbol{\tau}\, dt = d\mathbf{L}.$$

Thus the circular orbit and attached vector $\mathbf{L}$ rotate in a counterclockwise direction (looking into the B-field flux lines). The resulting precession is also shown in a top view of the circle traced out by the tip of the vector $\mathbf{L}$. From this geometry, it follows that the angle of rotation

$$d\phi = dL / L' = \tau\, dt / L \sin\theta,$$

and so the precessional angular velocity

$$\omega_p = \frac{d\phi}{dt} = \frac{\tau}{L \sin\theta} = \frac{\mu_e B \sin\theta}{L \sin\theta} = \frac{\mu_e B}{L}. \tag{2}$$

Now the magnetic moment of a circular current is given by

$$\mu = iA = i(\pi r^2),$$

where, in this case,

$$i = \frac{dQ}{dt} = ev = \frac{e\omega}{2\pi}.$$

Furthermore, the angular momentum of a particle can be expressed by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

or

$$|L| = mvr = mr^2\omega.$$

Substituting these expressions into Eq. (2),

$$\omega_p = eB/2m \quad \text{or} \quad \nu_p = eB/4\pi m. \tag{3}$$

Note that the precessional angular velocity is independent of the orientation of the particular current loop considered. Notice also that the overall effect is a rotation of the electronic structure (made up of many circulating currents) about the direction of the applied magnetic field.

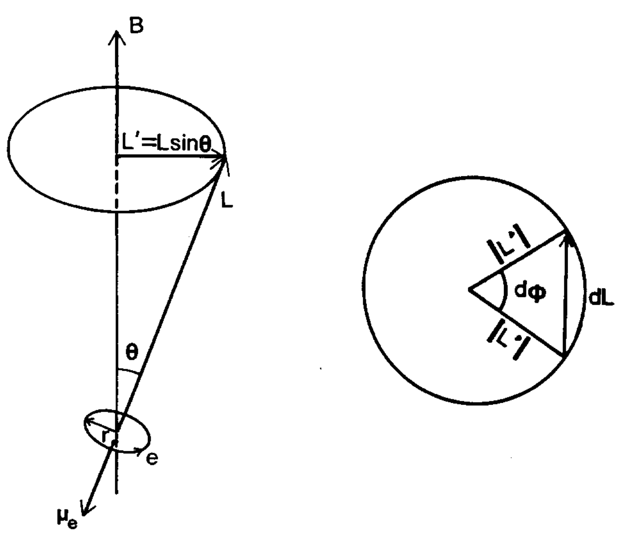

Fig. 1. Precession of the angular momentum vector about the direction of the applied magnetic field.

B. Circular birefringence

The optical rotation of the polarized light passing through the electronic structure can be understood as circular birefringence—the existence of different indices of refraction for left-circularly l- and right-circularly r- polarized light components. Each component traverses the sample with a different refractive index and therefore with a different speed. The end result consists of l and r components that are out of phase and whose superposition is linearly polarized light rotated relative to its original direction.

The situation is complicated by the fact that light of frequency ν is traversing an electronic system that is rotating with the Larmor frequency. Relative to the electronic structure, then, the l and r components of the light appear to rotate with frequencies of $\nu + \nu_L$ and $\nu - \nu_L$. Since in a dispersive medium the refractive index is frequency dependent, we may indicate the functional dependence of the two refractive indices by

$$n_l = n(\nu - \nu_L) \quad \text{and} \quad n_r = n(\nu + \nu_L).$$

These relations state that the two refractive indices for light traversing the sample in a magnetic field have the same values as the refractive indices for light of frequencies $\nu \pm \nu_L$ in the unmagnetized medium. Since the optical path difference for the l and r light is $(n_r - n_l)d$, after traversing a length d, we calculate (to good approximation) the quantity $n_r - n_l$ as follows:

$$n_r - n_l = n(\nu + \nu_L) - n(\nu - n_L)$$

$$= \left(n(\nu) + \frac{dn}{d\nu}\nu_L\right) - \left(n(\nu) - \frac{dn}{d\nu}\nu_L\right)$$

$$= 2\nu_L \frac{dn}{d\nu}.$$

Introducing the Larmor frequency found earlier and expressing the dispersion in terms of $dn/d\lambda$ from $|dn/d\nu| = (\lambda^2/c)dn/d\lambda$, we have

$$n_r - n_l = 2\left(\frac{eB}{4\pi m}\right)\left(\frac{\lambda^2}{c}\frac{dn}{d\lambda}\right).$$

Now consider the superposition of the l and r components on emerging from the sample. The phase change during traversal for each is

$$\phi_r = (n_r d/\lambda)2\pi \quad \text{and} \quad \phi_l = (n_l d/\lambda)2\pi.$$

Since the two phases are identical on entering the sample, Fig. 2 shows ϕ_l and ϕ_r as rotation from $\phi = 0$. The vector sum of the two electric fields on emerging is shown as E, with a net rotation of β from its initial value. In the parallelogram it is evident that

$$\phi_l - \beta = \phi_r + \beta,$$

or

$$\beta = \tfrac{1}{2}(\phi_l - \phi_r).$$

Thus

$$\beta = \frac{1}{2}\left(\frac{2\pi d}{\lambda}\right)(n_l - n_r) = \frac{\pi d}{\lambda}\left(\frac{eB}{2\pi m}\right)\left(\frac{\lambda^2}{c}\frac{dn}{d\lambda}\right),$$

$$\beta = \left(\frac{e}{2m}\frac{\lambda}{c}\frac{dn}{d\lambda}\right)Bd,$$

so that the Verdet constant, by comparison with Eq. (1), is

$$V = \frac{e}{2mc}\lambda\frac{dn}{d\lambda}. \qquad (4)$$

We note that the Verdet constant is proportional to both the wavelength of the light and to its dispersion in the medium. When the constants are evaluated and V is expressed in its standard units of min/(G cm), Eq. (4) becomes

$$V = 1.0083\lambda\left(\frac{dn}{d\lambda}\right).$$

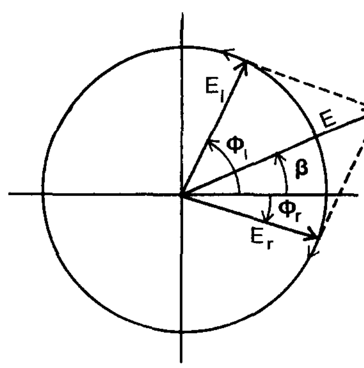

Fig. 2. Recomposition of left- and right-circularly polarized light into linearly polarized light after traversing the sample.

Table I. Experimental values of Faraday rotation.

	β(min)	For red (632.8 nm)	
B (kG)	CS_2 ($d = 1.99$ cm)	SF58 ($d = 2.67$ cm)	SF57 ($d = 2.73$ cm)
0	0	0	0
0.930	43	156	156
1.526	79	291	276
2.121	122	414	390
2.716	165	525	474
3.312	202	654	594
3.907	242	762	696
4.503	281	894	792
5.098	318	1020	900
5.694	361	1134	1014

III. EXPERIMENTAL

Both liquid (CS_2) and solid (flint glass) samples were used in the experiment. The samples were placed between the tapered poles of an electromagnet (Cenco model K) that provides fields up to around 6000 G. A central hole of diameter 6 mm drilled through the pole pieces, together with hollow mounting bolts, allowed He–Ne laser beams (both red at 632.8 nm and green at 543.5 nm) to pass through the samples along the field direction. A dichroic analyzer–polarizer pair capable of 0.1° accuracy was used. The polarizer was situated at the laser output to polarize the beam linearly, and the analyzer was placed in front of a silicon photodiode to detect the beam at the exit end. With the magnetic field off, the analyzer–polarizer pair is first set for extinction. The rotation of the analyzer required to reestablish extinction at various field strengths is then measured as the Faraday rotation angle β. The empty glass cell

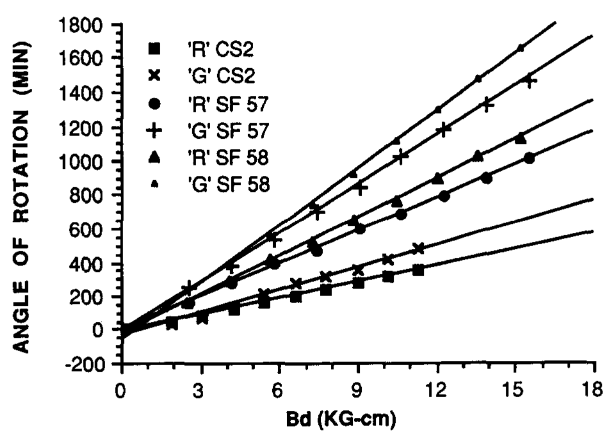

Fig. 3. Faraday rotation versus product of magnetic field and sample thickness for green laser light of 543.5 nm and red laser light of 632.8 nm in three samples.

Table II. Verdet constant and dispersion calculated from data.

	V [min/(G cm)]		$dn/d\lambda$ (nm^{-1})	
	543.5 nm	632.8 nm	543.5 nm	632.8 nm
CS_2	0.0447	0.0326	8.16×10^{-5}	5.11×10^{-5}
SF57	0.0956	0.0649	1.74×10^{-4}	1.02×10^{-4}
SF58	0.1123	0.0755	2.05×10^{-4}	1.18×10^{-4}

was also measured for its rotation as a function of field, and the results were used to correct the data for the cell filled with carbon disulfide.

The glass samples were SF57 and SF58 flints[11] with refractive indices at 587.6 nm of 1.847 and 1.918, respectively. The samples were cut in the form of rectangular solids, nominally $\frac{1}{2}\times\frac{1}{2}\times1$ in., with polished opposing faces along the long dimension.

IV. RESULTS

Sample data using the red He–Ne laser (632.8 nm) are given in Table I. Together with the data using the green He–Ne laser (543.5 nm), these data are plotted in Fig. 3: Faraday rotation angle β versus the product of magnetic field B and sample thickness d. According to Eq. (1), the slope of these lines, determined by linear regression, represents the Verdet constant in units of min/(G cm). We notice that the Faraday rotation is greater for the shorter wavelength, in spite of the proportionality indicated in Eq. (4). It is the larger dispersion at shorter wavelength that dominates the overall behavior. Measurement of the Verdet constant provides an indirect measurement of the dispersion, which can be calculated from Eq. (4). A summary of the results appears in Table II.

Major errors in experimental measurements derive from uncertainty in judging intensity minima and the uncertainty in the value of the magnetic field effective within the sample. We estimate average upper limit errors of 2% and 1%, respectively, so that values of Verdet constant and dispersion should have an accuracy of less than about 3%. Errors in reading minima can be reduced by taking as the correct analyzer setting an angular position intermediate between positions corresponding to equal photodiode outputs on either side of the minimum.

ACKNOWLEDGMENT

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A 0.7-mW magnetic heat engine

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A small magnetic heat engine is described that clearly demonstrates the principles involved and possible usefulness on a larger scale. It attracts considerable attention at open house demonstrations. Operating between 0 and ~4 °C, it produces a measurable output of 0.7 mW corresponding to 0.5 W kg^{-1} of working substance (gadolinium).

I. INTRODUCTION

The first patents on thermomagnetic devices began to appear 100 years ago.[1–3] Most of them concentrated on the conversion of heat to electrical energy. Their efficiency was discussed in detail by Brillouin and Iskenderian[4] and then reexamined by Elliot,[5] with emphasis on the use of gadolinium and its "room temperature" Curie point ($T_C = 293$ K). Meanwhile, Good ("Tom Tit")[6] had constructed a simple rotary magnetic device in which heat energy was used just to overcome the mechanical friction within the device. The theory of thermomagnetic engines was devel-